GENERALIZED SHAPIRO POLYNOMIALS AND PONS MATRICES*

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1°. Generalized Shapiro polynomials are given in the work [1] by the following recursive relations

$$
\begin{aligned}
P_{m+1,4j}(x) &= P_{m,2j}(x) + x^{2^m} P_{m,2j+1}(x), \\
P_{m+1,4j+1}(x) &= P_{m,2j}(x) - x^{2^m} P_{m,2j+1}(x), \\
P_{m+1,4j+2}(x) &= P_{m,2j+1}(x) + x^{2^m} P_{m,2j}(x), \\
P_{m+1,4j+3}(x) &= -P_{m,2j+1}(x) + x^{2^m} P_{m,2j}(x), \\
m = 1, 2, \ldots; &\quad j \in 0 : 2^{m-1} - 1,
\end{aligned} \tag{1}
$$

and the initial conditions

$$P_{1,0}(x) = 1 + x, \quad P_{1,1}(x) = 1 - x.$$

In this paper we investigate some of the properties of such polynomials.

2°. From (1), in particular, it follows that for $m = 1$ (and $j = 0$)

$$
\begin{aligned}
P_{2,0}(x) &= 1 + x + x^2 - x^3, \\
P_{2,1}(x) &= 1 + x - x^2 + x^3, \\
P_{2,2}(x) &= 1 - x + x^2 + x^3, \\
P_{2,3}(x) &= -1 + x + x^2 + x^3.
\end{aligned} \tag{2}
$$

It is clear that the polynomials $P_{m,k}(x)$ are defined for $k \in 0 : 2^m - 1$, their degree is $2^m - 1$ and all their coefficients are equal to ± 1. These polynomials are defined

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solely by the set of signs of their coefficients Therefore, formulas (2) can be written in the form

$$\begin{aligned} P_{2,0} &= (+++-), \\ P_{2,1} &= (++-+), \\ P_{2,2} &= (+-++), \\ P_{2,3} &= (-+++). \end{aligned}$$

Similarly we can present the polynomials $P_{3,k}(x)$, $k \in 0:7$. Basing on formulas (1) with $m = 2$ and $j \in 0:1$ we have

$$\begin{aligned} P_{3,0} &= (+++-++-+), \\ P_{3,1} &= (+++---+-), \\ P_{3,2} &= (++-++++-), \\ P_{3,3} &= (--+-+++-), \\ P_{3,4} &= (+-++-+++), \\ P_{3,5} &= (+-+++---), \\ P_{3,6} &= (-++++-++), \\ P_{3,7} &= (+---+-++). \end{aligned} \tag{3}$$

3°. From the definition of the generalized Shapiro polynomials it follows that for all complex z with $|z| = 1$, the following inequality holds:

$$|P_{m,2j}(z)|^2 + |P_{m,2j+1}(z)|^2 \leq 2^{m+1}.$$

Actually, a more refined result takes place, as noted in [1].

THEOREM 1. *For all $m \geq 1$, $j \in 0 : 2^{m-1} - 1$ and complex z with $|z| = 1$ the following identity holds*

$$|P_{m,2j}(z)|^2 + |P_{m,2j+1}(z)|^2 \equiv 2^{m+1}. \tag{4}$$

P r o o f. We use the following equality, which is true for all complex numbers z_1, z_2:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

For $m = 1$ $(j = 0)$ and $|z| = 1$ we have

$$|P_{1,0}(z)|^2 + |P_{1,1}(z)|^2 = |1 + z|^2 + |1 - z|^2 \equiv 4.$$

We now make the induction step from m to $m + 1$, assuming that $m \geq 1$.

The index $j \in 0 : 2^m - 1$ can be expressed in the form $j = 2j'$ or $j = 2j' + 1$, where $j' \in 0 : 2^{m-1}-1$. In the first case, basing on (1) and the inductive hypothesis, we have

$$\begin{aligned} |P_{m+1,2j}(z)|^2 + |P_{m+1,2j+1}(z)|^2 = |P_{m+1,4j'}(z)|^2 + |P_{m+1,4j'+1}(z)|^2 = \\ = \left|P_{m,2j'}(z) + z^{2^m} P_{m,2j'+1}(z)\right|^2 + \left|P_{m,2j'}(z) - z^{2^m} P_{m,2j'+1}(z)\right|^2 = \\ = 2\big(|P_{m,2j'}(z)|^2 + |P_{m,2j'+1}(z)|^2\big) \equiv 2^{m+2}. \end{aligned}$$

The case $j = 2j' + 1$ is considered in a similar manner.

The theorem is proved. □

4°. We select the first two equalities from (1), when $j = 0$:

$$\begin{aligned} P_{m+1,0}(x) &= P_{m,0}(x) + x^{2^m} P_{m,1}(x), \\ P_{m+1,1}(x) &= P_{m,0}(x) - x^{2^m} P_{m,1}(x), \\ m &= 1, 2, \ldots. \end{aligned}$$

We add the initial conditions

$$P_{1,0}(x) = 1 + x, \quad P_{1,1}(x) = 1 - x.$$

Recall (see [2]) that such recursive relations define the Shapiro polynomials. Therefore,

$$P_{m,0}(x) = P_m(x), \quad P_{m,1}(x) = Q_m(x).$$

It is known that the vectors of the coefficients of the Shapiro polynomials $P_m(x)$ and $Q_m(x)$ are orthogonal. Analogous property takes place for the generalized Shapiro polynomials.

Denote by $\mathbf{a}_{m,k}$ the vector of the coefficents of the polynomial $P_{m,k}(x)$.

THEOREM 2. *The vectors $\mathbf{a}_{m,k}$, $k \in 0 : 2^m - 1$, are pairwise orthogonal and $\|\mathbf{a}_{m,k}\|^2 = 2^m$.*

Proof. Recall that the coefficients of the Shapiro polynomials take values ± 1, therefore $\|\mathbf{a}_{m,k}\|^2 = 2^m$. We now verify orthogonality. When $m = 1$ it is obvious. We make the induction step from m to $m + 1$. For this purpose, we rewrite the relations (1) in the following form

$$\begin{aligned} \mathbf{a}_{m+1,4j} &= (\mathbf{a}_{m,2j}, \mathbf{a}_{m,2j+1}), \\ \mathbf{a}_{m+1,4j+1} &= (\mathbf{a}_{m,2j}, -\mathbf{a}_{m,2j+1}), \\ \mathbf{a}_{m+1,2j+2} &= (\mathbf{a}_{m,2j+1}, \mathbf{a}_{m,2j}), \\ \mathbf{a}_{m+1,4j+3} &= (-\mathbf{a}_{m,2j+1}, \mathbf{a}_{m,2j}). \end{aligned}$$

By the induction hypothesis, $\langle \mathbf{a}_{m,k}, \mathbf{a}_{m,k'} \rangle = 0$ for $k \neq k'$. Therefore,

$$\begin{aligned}
\langle \mathbf{a}_{m+1,4j}, \mathbf{a}_{m+1,4j+1} \rangle &= \langle \mathbf{a}_{m,2j}, \mathbf{a}_{m,2j} \rangle - \langle \mathbf{a}_{m,2j+1}, \mathbf{a}_{m,2j+1} \rangle = 0, \\
\langle \mathbf{a}_{m+1,4j}, \mathbf{a}_{m+1,4j+2} \rangle &= \langle \mathbf{a}_{m,2j}, \mathbf{a}_{m,2j+1} \rangle + \langle \mathbf{a}_{m,2j+1}, \mathbf{a}_{m,2j} \rangle = 0.
\end{aligned}$$

In a similar manner we see that

$$\langle \mathbf{a}_{m+1,4j+\sigma}, \mathbf{a}_{m+1,4j'+\sigma'} \rangle = 0 \quad \text{for } (j,\sigma) \neq (j',\sigma').$$

The theorem is proved. $\square$

5°. A square matrix $\mathcal{P}_m$ of size 2^m with rows

$$\mathbf{a}_{m,0}, \mathbf{a}_{m,1}, \ldots, \mathbf{a}_{m,2^m-1}$$

is referred to as PONS matrix [3]. The formula (3) gives the presentation of the matrix $\mathcal{P}_3$. The matrix $\mathcal{P}_4$ has the following form:

$$\mathcal{P}_4 = \begin{bmatrix}
+ & + & + & - & + & + & - & + & + & + & + & - & - & - & + & - \\
+ & + & + & - & + & + & - & + & - & - & - & + & + & + & - & + \\
+ & + & + & - & - & - & + & - & + & + & + & - & + & + & - & + \\
- & - & - & + & + & + & - & + & + & + & + & - & + & + & - & + \\
+ & + & - & + & + & + & + & - & - & - & + & - & + & + & + & - \\
+ & + & - & + & + & + & + & - & + & + & - & + & - & - & - & + \\
- & - & + & - & + & + & + & - & + & + & - & + & + & + & + & - \\
+ & + & - & + & - & - & - & + & + & + & - & + & + & + & + & - \\
+ & - & + & + & - & + & + & + & + & - & + & + & + & - & - & - \\
+ & - & + & + & - & + & + & + & - & + & - & - & - & + & + & + \\
+ & - & + & + & + & - & - & - & + & - & + & + & - & + & + & + \\
- & + & - & - & - & + & + & + & + & - & + & + & - & + & + & + \\
- & + & + & + & + & - & + & + & + & - & - & - & + & - & + & + \\
- & + & + & + & + & - & + & + & - & + & + & + & - & + & - & - \\
+ & - & - & - & + & - & + & + & - & + & + & + & + & - & + & + \\
- & + & + & + & - & + & - & - & - & + & + & + & + & - & + & +
\end{bmatrix}.$$

A row with index k of the $\mathcal{P}_m$ matrix can be seen as the values of the discrete signal

$$p_k(j) = \mathbf{a}_{m,k}(j), \quad k, j \in 0 : 2^m - 1.$$

According to theorem 2, the signals p_k are pairwise orthogonal and $\|p_k\|^2 = 2^m$ for all $k \in 0 : 2^m - 1$. This allows us to include the set of signals $\{p_k\}$ in the apparatus of discrete harmonic analysis.

On fig. 1 the plots of the functions $p_k(j)$ for $m = 3$ are given.

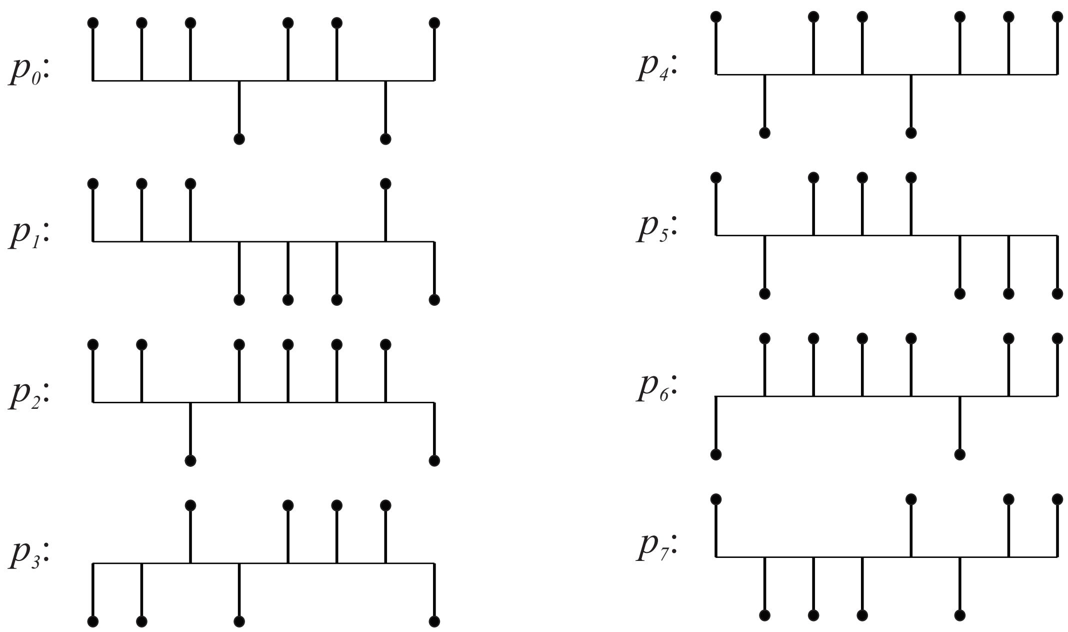

Figure 1

We formulate two hypotheses.

HYPOTHESIS 1. *The matrices $\mathcal{P}_m$ for all $m \geq 1$ are symmetric.*

For $m = 1, 2, 3$ the symmetry is verified directly. The symmetry of the matrices $\mathcal{P}_m$ up to $m = 10$ was checked using a computer program.

Let $n = 2^m$. By $k = (k_{m-2}, k_{m-3}, \ldots, k_0)_2$ we denote the binary expansion of the number $k \in 0 : \frac{n}{2} - 1$.

HYPOTHESIS 2. *For all $m \geq 2$ and $k, k' \in 0 : \frac{n}{2} - 1$ the following equality holds*

$$p_k(j) p_{k'}(j) = \alpha^m_{k,k'}\, p_{\frac{n}{2}+k}(j) p_{\frac{n}{2}+k'}(j), \quad j \in 0 : n-1, \tag{5}$$

where

$$\alpha^m_{k,k'} = (-1)^{k_{m-2}+k'_{m-2}}.$$

For $m = 3$ the matrix of coefficients $\{\alpha^3_{k,k'}\}$ is

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

The validity of the formula (5) for $m = 2, 3$ is verified directly, and for $m = 4, 5, \ldots, 10$ — with the aid of a computer program.

6°. Any real signal $x = x(j)$, $j \in 0 : n-1$ can be expanded in the PONS basis $\{p_k(j)\}_{k=0}^{n-1}$:

$$x(j) = \frac{1}{n} \sum_{k=0}^{n-1} \langle x, p_k \rangle \, p_k(j), \quad j \in 0 : n-1.$$

Consider the partial sums

$$x_\nu = \frac{1}{n} \sum_{k=0}^{\nu-1} \langle x, p_k \rangle \, p_k, \quad \nu \in 1 : n. \tag{6}$$

They can be represented as

$$x_\nu = D_{m,\nu} \, x, \tag{7}$$

where $D_{m,\nu}$ are square matrices of the form

$$D_{m,\nu} = \frac{1}{n} \sum_{k=0}^{\nu-1} \mathbf{a}_{m,k}^\top \, \mathbf{a}_{m,k}.$$

We used the definition of the signals p_k and the fact that $\mathbf{a}_{m,k}$ are row vectors. Note that the formula (6) is an equality of rows, and formula (7) is an equality of columns.

The norm of the matrix $D_{m,\nu}$ consistent with the uniform norm of a signal is referred to as Lebesgue constant. We denote it by $L_{m,\nu}$. By definition,

$$L_{m,\nu} = \max_{i \in 0:n-1} \sum_{k=0}^{n-1} \bigl| D_{m,\nu}[i,k] \bigr|.$$

We formulate two more hypotheses.

HYPOTHESIS 3. *The Lebesgue constants $L_{m,\nu}$ do not depend on m.*

The following table explains this hypothesis.

Table

m	$L_{m,\nu}, \quad \nu \in 1 : 2^m$															
2	1	1	$\frac{3}{2}$	1												
3	1	1	$\frac{3}{2}$	1	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	1								
4	1	1	$\frac{3}{2}$	1	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	1	$\frac{15}{8}$	$\frac{7}{4}$	$\frac{17}{8}$	$\frac{3}{2}$	$\frac{17}{8}$	$\frac{7}{4}$	$\frac{15}{8}$	1
5	1	1	$\frac{3}{2}$	1	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	1	$\frac{15}{8}$	$\frac{7}{4}$	$\frac{17}{8}$	$\frac{3}{2}$	$\frac{17}{8}$	$\frac{7}{4}$	$\frac{15}{8}$	1

HYPOTHESIS 4. *For $m \geq 2$ the following formulas hold*

$$L_{m,2^s} = 1, \quad s \in 0 : m; \tag{8}$$

$$L_{m,2^s+j} = L_{m,2^s-j} + \frac{j}{2^s}, \quad j \in 1 : 2^s - 1,\ s = 1, 2, \dots, m-1. \tag{9}$$

The validity of this hypothesis up to $m = 6$ was checked on a computer. The preliminary proof of hypothesis 4 that I have obtained is based on the hypotheses 1, 2 and 3.

The formulas (8), (9) allow us calculate the values of $L_{m,\nu}$ for all $\nu \in 1 : n$. According to (8), $L_{m,1} = L_{m,2} = 1$. When $s = 1$ and $j = 1$, by (9) we find $L_{m,3}$. For any following s we can calculate the values of $L_{m,\nu}$ for ν from 2^s to $2^{s+1} - 1$.

Fig. 2 shows the graph of $L_{m,\nu}$ as a function of ν for $m = 6$.

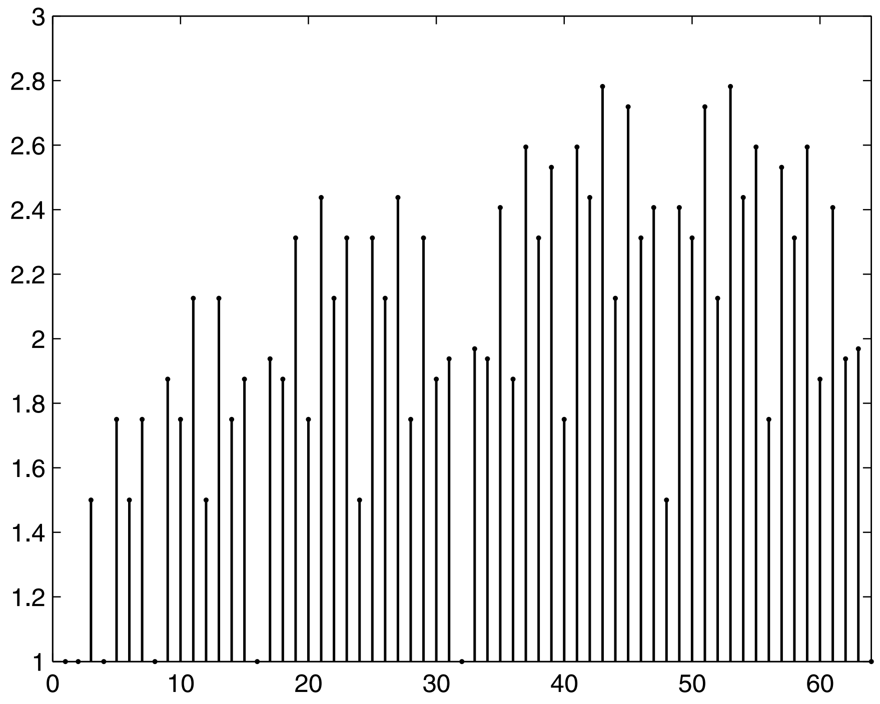

Figure 2

REFERENCES

1. Byrnes J. S. *Quadrature mirror filters, low crest factor arrays, functions achieving optimal uncertainty principle bounds, and complete orthonormal sequences – a unified approach* // Applied and Computational Harmonic Analysis. 1994. Vol. 1, P. 261–266.

2. Malozemov V. N., Stojanoska I. S. *Shapiro polynomials and Reed-Muller codes* // Seminar «DHA & CAGD». Selected papers. 26 November 2011 (`http://dha.spb.ru/reps11.shtml#1126`)

3. An M., Byrnes J., Moran W., Saffari B., Shapiro H. S., Tolimieri R. *PONS, Reed-Muller codes, and group algebras* // NATO Advanced Study Institute: Computational Noncommutative Algebra and Applications. Tuscany, Italy. July 2003. P. 155–197.