Sidelnikov inequality and Legendre polynomials

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Abstract. Extremal arrangements (in various meanings of "extremal") of points on a sphere have long been a subject of much interest (see [**1**] for a survey of early works).

One class of such extremal arrangements, consisting of "semidesigns" introduced in [**5**], is related to so-called Sidelnikov inequality [**2**]. In this paper we present a very simple proof of a discrete case of Sidelnikov inequality and show that it becomes equality on semidesigns.

1. INTRODUCTION. Let $\mathbb{R}^3$ denote Euclidean 3-space, $\langle x, y\rangle$ the scalar product, $\|x\|$ the norm $\sqrt{\langle x, x\rangle}$ and S^2 the unit sphere $\{x \in \mathbb{R}^3\colon \|x\| = 1\}$.

The following is a discrete case of Sidelnikov inequality in $\mathbb{R}^3$.

Theorem 1. *Let $x_1, \dots, x_m$ be any vectors in S^2 and k a positive integer. Then*

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\langle x_i, x_j\rangle^{2k} \geq \frac{m^2}{2k+1}. \tag{1}$$

Sidelnikov's proof of (1) is rather sophisticated. Goethals, Seidel [**3**] and Venkov [**4**] gave some other proofs. Our proof given below is based on Legendre polynomials.

2. SOME PROPERTIES OF LEGENDRE POLYNOMIALS. Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ can be defined by the conditions: $\deg P_n(t) = n$, $P_n(1) = 1$ and

$$\int_{-1}^{1} P_n(t)P_m(t)\,dt = 0, \quad n \neq m. \tag{2}$$

Any three consecutive polynomials P_{n-1}, P_n, P_{n+1} are connected by the following relation (see e.g. [**8**])

$$tP_n(t) = \frac{n+1}{2n+1}P_{n+1}(t) + \frac{n}{2n+1}P_{n-1}(t). \tag{3}$$

Also, we have $P_0(t) = 1$, $P_1(t) = t$.

We need the expansion of t^{2k} in terms of Legendre polynomials. For $k = 1$ we get $t^2 = tP_1(t) = \frac{2}{3}P_2(t) + \frac{1}{3}P_0(t)$. Multiply this equation by t and apply (3). We obtain

$$t^3 = \frac{2}{3}\Big[\frac{3}{5}P_3(t) + \frac{2}{5}P_1(t)\Big] + \frac{1}{3}P_1(t) = \frac{2}{5}P_3(t) + \frac{3}{5}P_1(t).$$

Multiplying by t again, we get

$$t^4 = \frac{8}{35}P_4(t) + \frac{4}{7}P_2(t) + \frac{1}{5}P_0(t).$$

Continuing this process ad infinitum, we obtain an expression of the form

$$t^{2k} = \sum_{l=0}^{k} c_l P_{2k-2l}(t), \tag{4}$$

with $c_l > 0$ for $l = 0, \dots, k$. Also, we can find explicitly the last coefficient c_k. To this end, we integrate (4) over $[-1, 1]$:

$$\int_{-1}^{1} t^{2k}\, dt = c_k \int_{-1}^{1} dt,$$

hence $c_k = 1/(2k+1)$.

In our proof of inequality (1), a decisive role is played by the so-called nonnegative definiteness of Legendre polynomials.

Let n be a positive integer. For any $n > 0$ there is the "addition formula"

$$P_n\big(\langle x, y\rangle\big) = \frac{4\pi}{2n+1} \sum_{s=1}^{2n+1} Y_{ns}(x) Y_{ns}(y), \quad x,\, y \in S^2. \tag{5}$$

Here $\{Y_{ns}\}$ are spherical functions of order n.

For any $X = \{x_1, \dots, x_m\} \subset S^2$ consider the sum

$$S_n(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} P_n\big(\langle x_i, x_j\rangle\big).$$

By (5) we can rewrite this as

$$S_n(X) = \tfrac{4\pi}{2n+1} \sum_{s=1}^{2n+1} \Big[\sum_{i=1}^{m} Y_{ns}(x_i)\Big]^2, \tag{6}$$

which yields $S_n(X) \geq 0$ — the above nonnegative definiteness.

3. PROOF OF THEOREM 1. Let again system $X = \{x_1, \dots, x_m\}$ be a subset of S^2. Taking $t = \langle x_i, x_j\rangle$ in (4) and summing over $i, j = 1, \dots, m$, we get

$$S := \sum_{i,j=1}^{m} \langle x_i, x_j\rangle^{2k} = \sum_{l=0}^{k} c_l S_{2k-2l}(X) \tag{7}$$

with $c_l > 0$ and $S_{2k-2l}(X) \geq 0$, hence $S \geq c_k S_0(X) = \frac{m^2}{2k+1}$. ∎

4. CONDITIONS FOR THE EQUALITY CASE IN (1)**.** Inequality (1) turns to equality on spherical semidesigns. Spherical semidesigns have been introduced in [**5**]. One of equivalent definitions is the following

Definition (Spherical Semidesign). A system $\Phi = \{\phi_1, \dots, \phi_m\}$ on S^2 is called spherical semidesign of order $2k$, iff

$$S_n(\Phi) := \sum_{i,j=1}^{m} P_n\big(\langle \phi_i, \phi_j\rangle\big) = 0 \quad \text{for}\, n = 2, 4, 6, \dots, 2k. \tag{8}$$

Let us compare this with the notion of spherical design introduced in [**6**]. One of the possible definitions for a spherical design of order p is the following: a system Φ is a spherical design of order p iff $S_n(\Phi) = 0$ for all $n = 1, 2, 3, \dots, p$ (see [**7**]).

Theorem 2. *Let $X = \{x_1, \dots, x_m\}$ be any system on S^2. The equation*

$$\sum_{i,j=1}^{m} \langle x_i, x_j \rangle^{2k} = \frac{m^2}{2k+1} \tag{9}$$

holds if and only if X is a spherical semidesign of order $2k$.

Proof. Let S denote the left-hand side of (9). The proof is based on representation (7). We rewrite it in the form

$$S = \sum_{l=0}^{k-1} c_l S_{2k-2l}(X) + \frac{m^2}{2k+1},$$

where $c_l > 0$, $l = 0, \dots, k-1$. If $S = m^2/(2k+1)$ then

$$\sum_{l=0}^{k-1} c_l S_{2k-2l}(X) = 0. \tag{10}$$

From $S_{2k-2l}(X) \geq 0$, $l = 0, \dots, k-1$, it follows that $S_{2k-2l}(X) = 0$ for all $l = = 0, \dots, k-1$, that is, $S_n(X) = 0$ for $n = 2, 4, \dots, 2k$. By the definition 1, the system X is a spherical semidesign of order $2k$.

Conversely, if X is a spherical semidesign of order $2k$, then (10) holds which yields $S = m^2/(2k+1)$. ∎

Example. Consider an icosahedron inscribed in S^2. Let $\{x_1, \dots, x_6, -x_1, \dots, -x_6\}$ be itsvertices. Then $\langle x_i, x_j \rangle = \pm 1/\sqrt{5}$ for $i \neq j$. Hence

$$\sum_{i,j=1}^{6} \langle x_i, x_j \rangle^4 = 6 \cdot 1 + 30 \cdot \tfrac{1}{25} = \tfrac{36}{5}.$$

For $k = 2$, $m = 6$ we have $m^2/(2k+1) = \frac{36}{5}$. By theorem 2 the system $X = = \{x_1, \dots, x_6\}$ is a spherical semidesign of order 4. The set of all 12 vertices of icosahedron is a spherical design of order 5.

If we take cube, octahedron or dodecahedron inscribed in S^2, then the set of vertices for each of these polyhedra is a spherical design of order 3. At the same time, their “halves” are spherical semidesigns of order 2.

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