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h-POLYHEDRAL SEPARABILITY OF TWO SETS

Suppose we have two finite sets in $\mathbb{R}^n$

$$A = \{a_i\}_{i=1}^m \quad \text{and} \quad B = \{b_j\}_{j=1}^k,$$

$$\operatorname{conv}(A) \cap B = \varnothing.$$

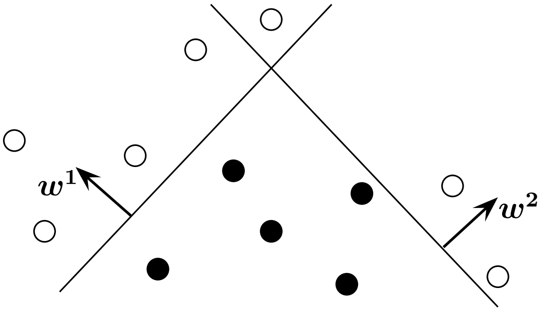
We will say that a convex hull of the set A and the set B are h-polyhedral separable if there exist h hyperplanes of a form

$$H_s = \{x \in \mathbb{R}^n \mid \langle w^s, x\rangle = \gamma_s,\ s \in 1:h,\}, \quad w^s \neq \mathbb{O},$$

such that the following inequalities hold:

$$\langle w^s, a_i\rangle - \gamma_s < 0 \quad \forall\, i \in 1:m,\ \forall\, s \in 1:h;$$
$$\langle w^s, b_j\rangle - \gamma_s > 0 \quad \text{for every} \quad j \in 1:k$$
$$\text{and some} \quad s \in 1:h.$$

The figure below shows an example of 2-polyhedral separability.



The problem can be formalized as follows:

$$\boldsymbol{F(G) := \frac{1}{m}\sum_{i=1}^{m}\max_{s\in 1:h}\big[\langle w^s, a_i\rangle - \gamma_s + c\big]_+ +}$$

$$\boldsymbol{+\frac{1}{k}\sum_{j=1}^{k}\min_{s\in 1:h}\big[-\langle w^s, b_j\rangle + \gamma_s + c\big]_+ \to \min_{G}.} \qquad (1)$$

Here G is a matrix of a size $h \times (n+1)$ with rows

$$\boldsymbol{g^s = (w^s, \gamma_s), \quad s \in 1 : h;}$$

$c > 0$ is a parameter, and $[u]_+ = \max\{0, u\}$. We'll say that the matrix G is *suitable* if all its vectors w^s are not equal to a zero vector.

The following statement holds.

THEOREM

The convex hull of the set A *and the set* B *are* h*-polyhedral separable if and only if there exists a suitable matrix* G_* *such that* $F(G_*) = 0$*.*

The problem (1) can be reduced to a finite number of linear programming problems. Indeed, denote $\Pi = \big\{ S = (s_1, \ldots, s_k) \mid s_j \in 1:h,\ j \in 1:k \big\}$.
Then

$$\inf_G F(G) = \min_{S\in\Pi} \inf_G \Big\{ \frac{1}{m} \sum_{i=1}^{m} \max_{s\in 1:h} \big[\langle w^s, a_i\rangle - \gamma_s + c\big]_+ + \\ + \frac{1}{k} \sum_{j=1}^{k} \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + c\big]_+ \Big\}. \tag{2}$$

From (2) we come to a system of extremal problems corresponding to different $S \in \Pi$:

$$\frac{1}{m}\sum_{i=1}^{m}\max_{s\in 1:h}\big[\langle w^s, a_i\rangle - \gamma_s + c\big]_+ +$$

$$\frac{1}{k}\sum_{j=1}^{k}\big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + c\big]_+ \to \inf_G .$$

Each of these problems is equivalent to a linear programming problem

$$\frac{1}{m}\sum_{i=1}^{m} p_i + \frac{1}{k}\sum_{j=1}^{k} q_j \to \inf, \tag{3}$$

$$-\langle a_i, w^s\rangle + \gamma_s + p_i \geq c, \quad i \in 1:m,\ s \in 1:h;$$

$$\langle b_j, w^{s_j}\rangle - \gamma_{s_j} + q_j \geq c, \quad j \in 1:k;$$

$$p_i \geq 0, \quad i \in 1:m; \quad q_j \geq 0, \quad j \in 1:k.$$

Each problem of a kind (3) has a solution because its set of plans is nonempty and its objective function is bounded below by zero.
Hence it follows that the problem (1) has a solution too.

Generally speaking, the optimal matrix is not necessarily suitable. In this context, the following statement is of an interest
(A. Astorino, M. Gaudioso, 2002).

THEOREM

Let $\boldsymbol{F}(\boldsymbol{G}_*) = \mathbf{0}$. In this case

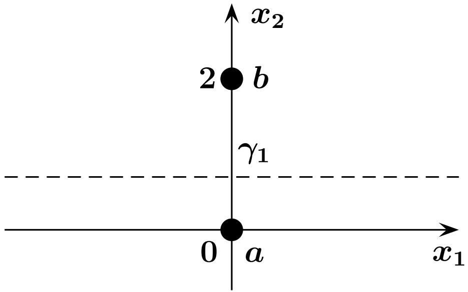
1. *not all vectors $\boldsymbol{w}_*^s$ of $\boldsymbol{G}_*$ are equal to zero;*
2. *if $\boldsymbol{w}_*^s = \mathbb{O}$ on a set $\boldsymbol{S} \subset \mathbf{1} : \boldsymbol{h}$ then the convex hull of the set $\boldsymbol{A}$ and the set $\boldsymbol{B}$ are $\big(\boldsymbol{h} - |\boldsymbol{S}|\big)$-polyhedral separable.*

By definition, $F(G) \geq 0$ for all matrices G. It is the feature of the problem (1) that there may exist a suitable matrix G_* that separates $co(A)$ from B in the sense of our definition, but at the same time $\boldsymbol{F(G_*) > 0}$.

EXAMPLE

Consider two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ on the plane $\mathbb{R}^{\mathbf{2}}$, each containing a single point, $\boldsymbol{a = (0, 0)}$ and $\boldsymbol{b = (0, 2)}$, respectively.
We'll naturally talk about 1–polyhedral separability.
The role of the parameter $\boldsymbol{c}$ will be clarified.

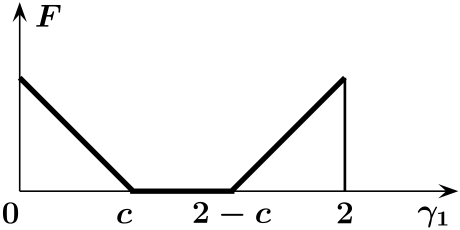
Let $w^1 = (0,1), \quad \gamma_1 \in (0,2)$.
In this case the line $x_2 = \gamma_1$ separates a from b.



At the same time

$$F(G) = [-\gamma_1 + c]_+ + [-2 + \gamma_1 + c]_+.$$

The figure shows a plot of $F(G)$ as a function of γ_1.



We can see that $F(G) = 0$ for pairs (w^1, γ_1) with $\gamma_1 \in [c, 2 - c]$.
When $\gamma_1 \in (0, c) \cup (2 - c, 2)$, pairs (w^1, γ_1) still separate a from b, but $F(G) > 0$.

Exhaustive search of elements S of a set Π generates a large number (h^k) of linear programming problems. Therefore, local methods are of interest.
Let us describe the concept of a method of gradient type.

Denote

$$f(v,u) = \langle v,u\rangle + c,$$

$$\hat{a}_i = \begin{pmatrix} a_i \\ -1 \end{pmatrix}, \quad \check{b}_j = \begin{pmatrix} -b_j \\ 1 \end{pmatrix},$$

$$\varphi_i(G) = \max_{s\in 1:h}\big[f(g^s,\hat{a}_i)\big]_+, \quad \psi_j(G) = \min_{s\in 1:h}\big[f(g^s,\check{b}_j)\big]_+,$$

and rewrite the problem (1) in a compact form

$$F(G) := \frac{1}{m}\sum_{i=1}^{m}\varphi_i(G) + \frac{1}{k}\sum_{j=1}^{k}\psi_j(G) \to \min_G .$$

The matrix function $F(G)$ is differentiable in directions, and an explicit formula for the derivative $F'(G, V)$ can be written. To minimize $F(G)$, the following procedure will be used.
Let G_k be the k-th approximation. Consider an auxiliary problem that determines the direction of a descent:

$$F'(G_k, V) \to \min_{V \in \Omega}, \tag{4}$$

where Ω is a set of matrices whose elements are all bounded in absolute value by $K > 0$. The problem (4) can be reduced to a small number of LP problems. It has a solution. We denote it V_k.

If

$$F'(G_k, V_k) \geqslant 0,$$

then G_k is a local minimum of $F(G)$. Calculations stop. Otherwise, proceed to the next approximation

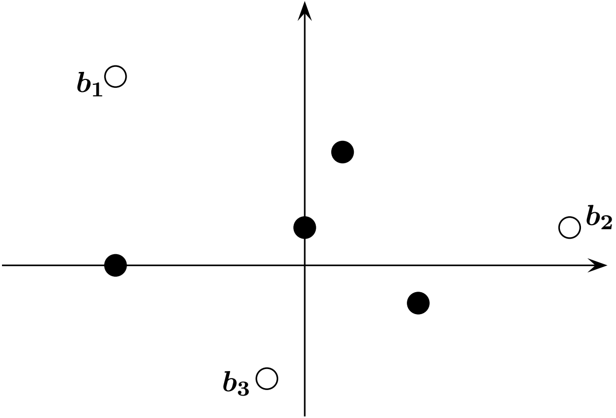
$$G_{k+1} = G_k + t_k V_k,$$

where a step $t_k > 0$ provides the required decrease of $F(G)$.

EXAMPLE

Consider the problem of $\mathbf{3}$-polyhedral separability for different values of the parameter $\boldsymbol{c}$. Let the sets $\boldsymbol{A}$ and $\boldsymbol{B}$ on a plane be

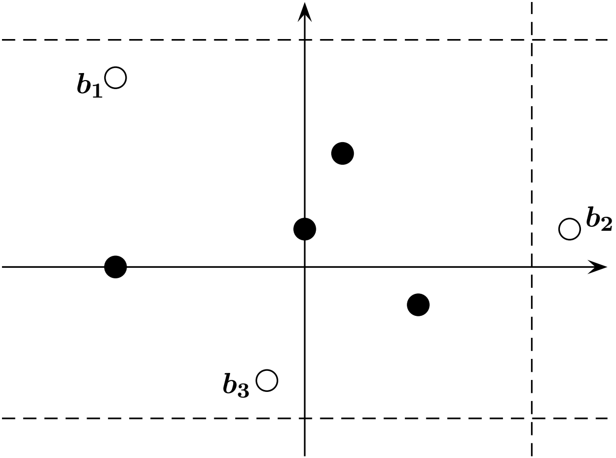
$$\boldsymbol{a_1 = (-3, 0),\ a_2 = (1, 2),\ a_3 = (2, -1),\ a_4 = (0, 1);}$$
$$\boldsymbol{b_1 = (-3, 3),\ b_2 = (4, 1),\ b_3 = (-1, -3).}$$



b_1
b_2
b_3

At first, let's solve the problem of 3-polyhedral separability with $c = 0$. The initial approximation is

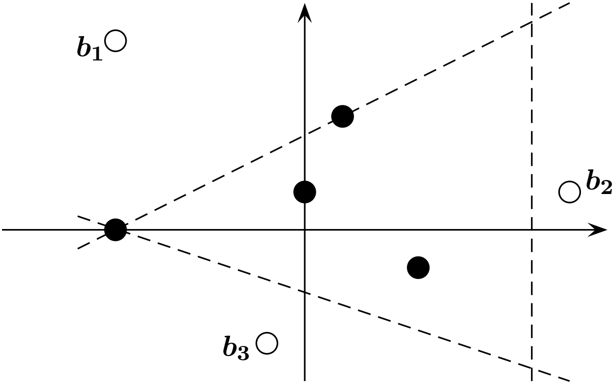
$$G_0 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & -1 & 4 \end{pmatrix}, \quad F(G_0) = \frac{2}{3}.$$



At the first step we obtain the matrix

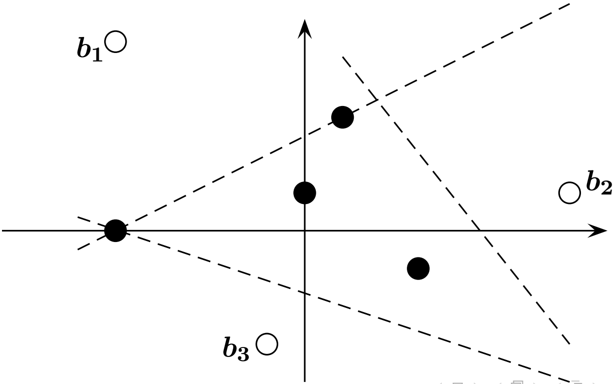
$$G_1 = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 3 \\ -1 & -2 & 3 \end{pmatrix},$$

which is the solution of the problem $(F(G_1) = 0)$.

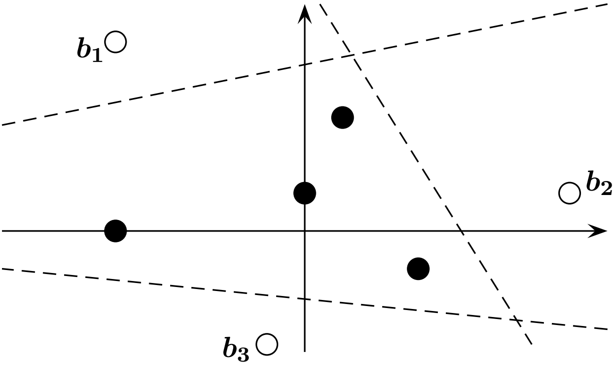


Let's change the value of the parameter c in this problem. Let's put $c = 1$ instead of $c = 0$. Take the same matrix G_0 as an initial approximation, $\boldsymbol{F(G_0) = \frac{4}{3}}$. We obtain

$$\boldsymbol{G_1 = \begin{pmatrix} 1.3 & 0.8 & 3.6 \\ -1 & 2 & 3 \\ -1 & -2 & 3 \end{pmatrix}, \quad F(G_1) = 0.5,}$$



$$G_2 = \begin{pmatrix} 1.3 & 0.8 & 3.6 \\ -0.6 & 1.6 & 3.4 \\ -0.6 & -2 & 3.4 \end{pmatrix}, \quad F(G_2) = 0.075.$$



For the matrix G_2, the inequalities from the definition of h-polyhedral separability hold. Therefore, the suitable matrix G_2 provides 3–polyhedral separability (although $F(G_2) > 0$).

Once again, change the value of the parameter c.
Put $c = \frac{1}{2}$, then $F(G_0) = 1$. Further,

$$G_1 = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 3 \\ -1 & -2 & 3 \end{pmatrix}.$$

The matrix G_1 is the same as for $c = 0$, but the value of the objective function is $F(G_1) = \frac{1}{4}$. At the next step we have

$$G_2 = \begin{pmatrix} 1 & 0 & 3 \\ -0.7 & 1.7 & 3.3 \\ -0.7 & -2 & 3.3 \end{pmatrix}, \quad F(G_2) = 0$$