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NONLINEAR CHEBYSHEV APPROXIMATIONS AND NONSMOOTH OPTIMIZATION

Consider the problem

$$\varphi(x) := \max_{t \in [a,b]} \left| f(x,t) \right| \to \inf_{x \in \mathbb{R}^n} . \qquad (1)$$

Fix $\hat{x} \in \mathbb{R}^n$ and denote by $a \leqslant t_1 < t_2 < \cdots < t_r \leqslant b$ the points that satisfy to

$$\left| f(\hat{x}, t_k) \right| = \varphi(\hat{x}), \quad k \in 1 : r.$$

THEOREM (P. L. Chebyshev, 1859)

The value $\varphi(\hat{x})$ is not brought to its smallest value if the system of equations

$$\sum_{k=1}^{r} f'_x(\hat{x}, t_k) \lambda_k = \mathbb{O}$$

has only a zero solution $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$.

(In other words, if the vectors $f'_x(\hat{x}, t_k)$, $k \in 1 : r$, are linearly independent.)

A modern formulation of Chebyshev's theorem:

THEOREM

If a point $\hat{x} \in \mathbb{R}^n$ *is a solution of a problem* (1) *then there exist nonnegative numbers* $\alpha_1, \alpha_2, \ldots, \alpha_r$ *whose sum equals to one, such that*

$$\sum_{k=1}^{r} \xi_k \alpha_k f'_x(\hat{x}, t_k) = \mathbb{O},$$

where $\xi_k = \operatorname{sign} f(\hat{x}, t_k)$ *and* t_k *are the points of the maximal deviation.*

The theorem's conditions are sufficient in a linear case when

$$f(x,t) = \sum_{i=1}^{n} x_i u_i(t) - u_0(t).$$

Denote by $\mathcal{P}_n$ a set of algebraic polynomials of order not greater than n and consider the extremal problem

$$\max_{t\in[-1,1]}\bigl|P_n(t)\bigr| \to \inf, \tag{2}$$

where the infimum is taken over all polynomials $P_n \in \mathcal{P}_n$ with the highest coefficient equal to one.

THEOREM (P. L. Chebyshev)

The only solution of the problem (2) *is a polynomial*

$$P_n^*(t) = \frac{1}{2^{n-1}}\cos\bigl(n(\arccos t)\bigr).$$

Chebyshev polynomials

$$T_n(t) = \cos(n \arccos t)$$

have many extremal properties. We'll remind one of them.

Consider the extremal problem

$$\begin{gathered} P_n(b) \to \sup, \\ \left|P_n(t)\right| \leqslant 1, \quad t \in [-1, 1], \end{gathered} \tag{3}$$

where $b > 1$ is a fixed point.

THEOREM (P. L. Chebyshev)

The only solution of the problem (3) *is a polynomial* $T_n(t)$.

Note that the solution doesn't depend on b.

Now we turn to a generalization of the problem (3):

$$\begin{gathered} P_n(b) \to \sup \\ \left|P_n(t)\right| \leqslant 1, \quad t \in [-1,1]; \\ P_n(a) = A; \quad x_n \geqslant 0, \end{gathered} \tag{4}$$

where x_n is the highest coefficient of the polynomial $P_n(t)$, and $a < -1$ and $b > 1$. Let, for the sake of determinancy, n be an odd number, $n \geqslant 3$.
We put $A_0 = T_n(a)$, $A_1 = T_{n-1}(a)$.

THEOREM

Given $A \in (A_0, A_1)$, a solution of the problem (4) exists and is unique. A polynomial $P_n^*(t)$ satisfying to the constraints of the problem (4) is its solution if and only if there exist n points $-1 \leqslant t_1 < t_2 < \cdots < t_n \leqslant 1$, such that

$$P_n^*(t_k) = (-1)^{n-k}, \quad k \in 1 : n.$$

We denote by $\mathcal{H}_n^n$ a collection of fractions $H(t)$ whose numerator and denominator are algebraic polynomials of order not greater than n.

THE THIRD ZOLOTAREV'S PROBLEM (1877):

$$\max_{t\in[-1,1]} \left|H(t)\right| \to \inf, \qquad (5)$$

where the infimum is taken over all fractions $H \in \mathcal{H}_n^n$ satisfying to the condition

$$\left|H(t)\right| \geqslant 1 \quad \text{for} \quad |t| \geqslant \frac{1}{\tau}, \quad \tau \in (0,1).$$

The problem (5) has two solutions that differ in sign only. A solution satisfying to the additional condition $H(\frac{1}{\tau}) = 1$ is referred to as Zolotarev fraction.

Zolotarev fractions, in particular, help to solve the problem of optimal ADI parameters:

$$\max_{t\in[\gamma,1]}\left|\prod_{j=1}^{n}\frac{t-r_j}{t+r_j}\right| \to \inf_{\{r_j\}},$$

where $\gamma \in (0,1)$ is a parameter.
In 1933, W. Couer pointed out that Zolotarev fractions have important applications in problems of electrical filters synthesis. Three-band filters were mathematically examined by R. A.-R. Amer (1964).
A great contribution into investigating all four Zolotarev's problems was paid by N. I. Akhiezer (1901–1980).

Let a finite system of pairwise disjoint intervals $[c_i, d_i]$, $i \in I$, be given on a ray $(0, +\infty)$. Denote

$$D_0 = \bigcup_{i \in I_0} [c_i, d_i], \quad D_1 = \bigcup_{i \in I_1} [c_i, d_i]; \quad D_0 \cap D_1 = \varnothing.$$

Consider a collection of functions

$$H(X, t) = g(t) \tfrac{P(A,t)}{Q(B,t)},$$

where $X = (A, B)$, and $P(A, t)$ and $Q(B, t)$ are algebraic polynomials of order n and m, respectively. The extremal problem

$$\varphi(X) := \max\left\{ \max_{t \in D_0} \left| H(X, t) \right|, \max_{t \in D_1} \frac{1}{\left| H(X,t) \right|} \right\} \to \inf_{\{X\}}, \quad (6)$$

is called a problem of multi-band electrical filter synthesis.

The problem (6) is multiextremal, however it is possible to describe all its local solutions. The sets

$$\Omega(\sigma) = \bigcap_{i \in I_0} \{X = (A, B) \mid \sigma_i Q(B, t) > 0, t \in [c_i, d_i]\} \cap$$
$$\cap \bigcap_{i \in I_1} \{X = (A, B) \mid \sigma_i P(A, t) > 0, t \in [c_i, d_i]\},$$

where $\sigma_i = \pm 1$, are referred to as sign classes. The problem (6) is reduced to a problem of minimization over sign classes:

$$\max_{X \in \Omega(\sigma)} \varphi(X) \to \min_{\{\sigma\}}.$$

It is ascertained (V. N. Malozemov, 1979) that the problem of minimization over a non-empty sign class has a unique solution which admits an alternance characterization.

Tikhomirov functions

A perfect spline is a function of a form

$$T(t) = \frac{t^r}{r!} + \sum_{k=0}^{r-1} c_k t^k + \frac{2}{r!} \sum_{i=1}^{n} (-1)^i (t - x_i)_+.$$

Here $-1 < x_1 < x_2 < \cdots < x_n < 1$. The following property is characteristic to a perfect spline:

$$\left|T^{(r)}(t)\right| \equiv 1 \quad \text{on all intervals} \quad (x_i, x_{i+1}).$$

V. M. Tikhomirov investigated the following extremal problem:

$$\max_{t\in[-1,1]} \left|T(t)\right| \to \inf_{\{c_k, x_i\}}. \tag{7}$$

THEOREM (V. M. Tikhomirov, 1969)

A perfect spline that has the least deviation from zero exists and is unique.

I refer to the solution of the problem (7) as *Tikhomirov function*. Tikhomirov function $T_{r,\,n}(t)$ is characterized by the existence of a full alternance, that is, points

$$-1 = t_0 < t_1 < \cdots < t_{r+m} = 1,$$

in which $T_{r,\,n}(t_j) = (-1)^{r+j}\|T_{r,\,n}\|$, $j \in 0 : r+n$. The problem (7) can be widely generalized. The simplest generalization is gained if we take these constraints:

$$T^{(s)}(-1) = T^{(s)}(1) = 0, \quad s \in 0 : r-1.$$

A monospline is a function of a form

$$G(t) = t^{r+1} - \sum_{k=0}^{r} c_k t^k - \sum_{i=1}^{n} a_i (t - x_i)_+^r .$$

Monosplines appear in the analysis of quadrature rules

$$\int_0^1 f(x)dx = \tfrac{1}{r+1} \sum_{i=1}^{n} a_i f(x_i) + R(f),$$

where $0 \leqslant x_1 < x_2 < \cdots < x_n \leqslant 1$. The best quadrature rule on the class $W_1^{r+1}M$ is gained as a result of solving the extremal problem

$$\sup_{f \in W_1^{r+1}M} \left| R(t) \right| \to \inf_{\{a_i, x_i\}} . \qquad (8)$$

The problem (8) is reduced to the problem of a monospline with the least deviation from zero:

$$\max_{t\in[0,1]}\left|G(t)\right| \to \inf_{\{a_i, x_i\}}$$

subject to $G^{(s)}(0) = G^{(s)}(1) = 0$, $s \in 0 : r$.

THEOREM (V. N. Malozemov and A. B. Pevnyi, 1980)

The optimal quadrature rule on the class $W_1^{r+1}M$ *exists and is unique. Moreover,* $0 < x_1 < x_2 < \cdots < x_n < 1$*, and all coefficients* a_i *are positive.*

To find out the parameters $\{a_i, x_i\}$ of the optimal quadrature rule one has to solve a nonlinear system of equations of order $4n - r + 1$.

Numeric methods

Efficient numeric methods (in particular, linearization methods) are developed for approximate solving of discrete Chebyshev problems of a kind

$$\varphi(x) := \max_{i \in I} \left| f_i(x) \right| \to \inf_{x \in \mathbb{R}^n} . \tag{9}$$

We will describe a concept of the linearization method by B. N. Pshenichnyi.
Suppose we have the k-th approximation $x_k \in \mathbb{R}^n$. Consider an auxiliary problem that determines the direction of a descent:

$$\max_{i \in I} \left| f_i(x_k) + \left\langle f_i'(x_k), h \right\rangle \right| + \frac{1}{2} \|h\|^2 \to \min_{h \in \mathbb{R}^n} . \tag{10}$$

The problem (10) is reduced to a quadratic programming problem. It has a unique solution; we denote it by h_k.

If $h_k = \mathbb{O}$ then x_k is a stationary point. Calculations stop. Otherwise we proceed to the next approximation

$$x_{k+1} = x_k + t_k h_k,$$

where $t_k > 0$ is a step providing a required decrease of the objective function $\varphi(x)$.
Pshenichnyi's linearization method has a quadratic degree of convergence in case when the solution of the problem (9) has a full alternance (V. A. Daugavet and V. N. Malozemov, 1981).
For solving continuous Chebyshev problems, a method of alignment of maxima is used. This method also has a quadratic degree of convergence in a neighborhood of a point having a full alternance.

Consider the extremal problem

$$\varphi(x) := \max_{t\in[a,b]} \bigl| f(x,t) \bigr| \to \inf$$

subject to

$$f(x, \tau_j) = 0, \quad j \in 1 : m. \tag{11}$$

Let $x^* \in \mathbb{R}^n$ satisfy to the constraints (11). Suppose that the functions

$$u_k(t) = f'_{x_k}(x^*, t), \quad k \in 1 : n,$$

form a Chebyshev system on $[a, b]$. In this case, the following statement holds (V. A. Daugavet and V. N. Malozemov, 1977).

THEOREM

A point x^* *is a point of a strict local minimum of the function* $\varphi(x)$ *subject to the constraints* (11) *if and only if there exist points of the maximal deviation*

$$a \leqslant t_0 < t_1 < \cdots < t_{n-m} \leqslant b$$

such that

$$f(x^*, t_i) = (-1)^{s_i+1} f(x^*, t_{i-1}), \quad i \in 1 : n-m,$$

where s_i *is the number of interpolation nodes* τ_j *lying inside the interval* (t_{i-1}, t_i).