GLOBAL REGULARITY IN MATHEMATICAL PROGRAMMING*

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We present an attempt to reformulate and to complete the papers [1, 2].

1°. We consider the mathematical programming problem

$$\begin{gathered} f(x) \to \inf\,, \\ g_i(x) \le 0\,, \quad i \in 1:s\,; \\ x \in P\,. \end{gathered} \tag{1}$$

We assume that $P \subset \mathbb{R}^n$ is an arbitrary nonempty set (possibly discrete) and that $f, g_1, \ldots, g_s$ are arbitrary finite functions defined on P. We define

$$X = \big\{ x \in P \mid g_i(x) \le 0,\ i \in 1:s \big\}\,,$$

$$f^* = \inf \big\{ f(x) \mid x \in X \big\}.$$

We introduce the Lagrangian

$$L(x,y) = f(x) + \sum_{i=1}^{s} y_i\, g_i(x)$$

and define the dual problem

$$\varphi(y) := \inf \big\{ L(x,y) \mid x \in P \big\} \to \sup_{y \in \mathbb{R}^s_+}\,. \tag{2}$$

Here $\mathbb{R}^s_+$ is the set of non-negative vectors $y = (y_1, \ldots, y_s)$. We remark that the Lagrangian function $L(x,y)$ for a fixed x is affine as a function of y and, consequently, the dual objective function $\varphi(y)$ is *concave* in $\mathbb{R}^s_+$. We denote

$$\varphi^* = \sup \big\{ \varphi(y) \mid y \in \mathbb{R}^s_+ \big\}.$$

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In connection to (1), (2) we study the parametric auxiliary problem

$$\begin{gathered} f(x) \to \inf\,, \\ g_i(x) \le v_i\,, \quad i \in 1:s\,; \\ x \in P\,. \end{gathered} \tag{3}$$

We denote

$$X(v) = \big\{x \in P \mid g_i(x) \le v_i\,,\ i \in 1:s\big\}.$$

Function

$$F(v) = \inf\big\{f(x) \mid x \in X(v)\big\}, \quad v \in \mathbb{R}^s,$$

is the *sensitivity function* for problem (1).

The initial problem (1) coincides with (3) for $v = \mathbb{O}$. Function $F(v)$ reflects the characteristics of such inclusion. Moreover the sensitivity function is related to the objective function of the dual problem (2). The following property holds:

LEMMA. *For all $y \in \mathbb{R}^s_+$ function $\varphi(y)$ can be expressed as:*

$$\varphi(y) = \inf\big\{F(v) + \langle y, v\rangle \mid v \in \mathbb{R}^s\big\}.$$

The sensitivity function plays an important role in investigating both problems (1) and (2).

2°. Relatively to problem (1) we make the natural assumptions:

$$X \ne \varnothing\,, \qquad f^* > -\infty. \tag{4}$$

From the definition it follows that $f^* \ge \varphi^*$. A first question which has to be addressed is to investigate when the *duality relationship* $f^* = \varphi^*$ holds. To this aim, we have the following theorem.

THEOREM 1. *The duality condition holds if and only if the ε-subdifferential of the sensitivity function at point zero, $\partial_\varepsilon F(\mathbb{O})$, is not empty for all $\varepsilon > 0$.*

3°. The condition

$$\partial F(\mathbb{O}) \ne \varnothing \tag{5}$$

is defined as *global regularity* of problem (1). Substantially it characterizes the "regularity" of the inclusion of problem (1) into the family of parametric problems defined by (3).

THEOREM 2. *The global regularity condition is satisfied if and only if $f^* = \varphi^*$ and the dual problem (2) has solution.*

From the proof of the theorem we observe that:

- any point in $-\partial F(\mathbb{O})$ is a solution to (2);
- whenever condition $f^* = \varphi^*$ is satisfied, any solution of the dual belongs to the set $-\partial F(\mathbb{O})$.

4°. We provide now a sufficient condition for ensuring global regularity of problem (1).

THEOREM 3. *Let $f^* = \varphi^*$ and assume in addition that Slater condition is satisfied, that is there exists a point $z \in P$, such that $g_i(z) < 0$ for all $i \in 1:s$. Then $\partial F(\mathbb{O}) \neq \varnothing$.*

5°. We assume that the global regularity condition (5) is satisfied for problem (1). Consider any dual solution y^* with the related Lagrangian problem

$$L(x, y^*) \to \inf_{x \in P}. \tag{6}$$

Its extremal value satisfies the condition $\varphi(y^*) = \varphi^*$.

THEOREM 4 (Lagrange principle). *Any solution x^* of the problem* (1) *satisfying the global regularity condition solves problem* (6). *Moreover the complementarity condition*

$$y_i^* \, g_i(x^*) = 0 \,, \quad i \in 1:s \,. \tag{7}$$

is satisfied.

Now we present an example where both problems (1) and (6) have unique solution, but they are different from each other.

EXAMPLE. Consider the extremal problem

$$\begin{aligned} f(x) &:= x - 1 \to \inf \,, \\ g_1(x) &:= x^2 - 1 \leq 0 \,, \\ g_2(x) &:= -x^2 + 1 \leq 0 \,, \\ P &= \mathbb{R}_+ \,. \end{aligned}$$

It has optimal solution $x^* = 1$, with corresponding optimal value $f^* = 0$.

We consider the Lagrangian function

$$\begin{aligned} L(x, y) &= x - 1 + y_1 \, (x^2 - 1) + y_2 \, (-x^2 + 1) = \\ &= x^2 \, (y_1 - y_2) + x - (y_1 - y_2 + 1) \,. \end{aligned}$$

For the dual objective function it holds

$$\varphi(y) := \inf \big\{ L(x, y) \mid x \geq 0 \big\} = \begin{cases} -(y_1 - y_2 + 1) & \text{for } y_1 - y_2 \geq 0 \,, \\ -\infty & \text{for } y_1 - y_2 < 0 \,. \end{cases}$$

Here it is $y \in \mathbb{R}_+^2$. Obviously it holds $\varphi^* = -1$.

Any $y^* \geq \mathbb{O}$ with $y_1^* = y_2^*$ solves the dual. For any such vector the Lagrangian problem has the form

$$L(x, y^*) := x - 1 \to \inf_{x \geq 0}.$$

Its unique solution is $\hat{x} = 0$, which is different from the optimal solution $x^* = 1$ of the initial problem.

Note that in the example the duality condition does not hold, since $f^* = 0$ and $\varphi^* = -1$.

6°. We discuss now the sufficient condition for global optimality. A pair $\{x^*, y^*\}$, where $x^* \in P$, $y^* \in \mathbb{R}^s_+$, is said to be *globally optimal* [3, p. 144], if

$$(\alpha) \qquad L(x^*, y^*) = \min_{x \in P} L(x, y^*)\,;$$

$$(\beta) \qquad y_i^*\, g_i(x^*) = 0\,, \qquad i \in 1:s\,;$$

$$(\gamma) \qquad g_i(x^*) \leq 0\,, \qquad i \in 1:s\,.$$

THEOREM 5. *If $\{x^*, y^*\}$ is a globally optimal pair then x^* solves problem* (1), *y^* solves problem* (2) *and it is $f(x^*) = \varphi(y^*)$.*

P r o o f. Condition (γ) implies $x^* \in X$. For any $x \in X$ from (β) and (α) it follows

$$f(x^*) = f(x^*) + \sum_{i=1}^{s} y_i^*\, g_i(x^*) = L(x^*, y^*) \leq L(x, y^*) =$$
$$= f(x) + \sum_{i=1}^{s} y_i^*\, g_i(x) \leq f(x)\,.$$

Such condition ensures that x^* solves problem (1) and it is $f(x^*) = f^*$. Moreover

$$\varphi^* \geq \varphi(y^*) = \inf\big\{L(x, y^*) \mid x \in P\big\} = L(x^*, y^*) =$$
$$= f(x^*) + \sum_{i=1}^{s} y_i^*\, g_i(x^*) = f(x^*) = f^* \geq \varphi^*,$$

which implies both $\varphi(y^*) = \varphi^*$, $f(x^*) = \varphi(y^*)$ and the proof is complete. □

THEOREM 6. *The globally optimal pair $\{x^*, y^*\}$ is a saddle point of the Lagrangian function, that is for all $x \in P$ and $y \in \mathbb{R}^s_+$ the following holds*

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*)\,. \tag{8}$$

P r o o f. For all $y \in \mathbb{R}^s_+$ we have

$$L(x^*, y) = f(x^*) + \sum_{i=1}^{s} y_i\, g_i(x^*) \le f(x^*)\,,$$

thus

$$\sup\left\{L(x^*, y) \mid y \in \mathbb{R}^s_+\right\} \le f(x^*)\,.$$

Taking into account $f(x^*) = \varphi(y^*)$, we have

$$L(x^*, y^*) \le \sup\left\{L(x^*, y) \mid y \in \mathbb{R}^s_+\right\} \le f(x^*) =$$

$$= \varphi(y^*) = \inf\left\{L(x, y^*) \mid x \in P\right\} \le L(x^*, y^*)\,.$$

Consequently we obtain

$$\sup\left\{L(x^*, y) \mid y \in \mathbb{R}^s_+\right\} = L(x^*, y^*) = \inf\left\{L(x, y^*) \mid x \in P\right\},$$

which is equivalent to (8), and the proof is complete. □

7°. As a simple example of application of the general results previously stated we consider the linear programming problem

$$\begin{aligned} f(x) &:= \langle c, x\rangle \to \inf\,, \\ A\,x &\le b\,, \\ P &= \mathbb{R}^n_+\,. \end{aligned} \tag{9}$$

The dual of such linear program is

$$\begin{aligned} \psi(u) &:= \langle b, u\rangle \to \sup\,, \\ u\,A &\le c\,, \quad u \le \mathbb{O}\,. \end{aligned} \tag{10}$$

Let U be the set of feasible solutions to (10). Under hypothesis (4) both problems (9) and (10) have solution and it is

$$f^* := \min_{x \in X} f(x) = \max_{u \in U} \psi(u) \;=:\; \psi^*.$$

Now we study the connection between (10) and (2). We have

$$\varphi(y) := \inf\left\{\langle c, x\rangle + \langle y, A\,x - b\rangle \mid x \in \mathbb{R}^n_+\right\} =$$

$$= -\langle b, y\rangle + \inf\left\{\langle c + y\,A, x\rangle \mid x \in \mathbb{R}^n_+\right\}.$$

Obviously it is

$$\inf\left\{\langle c+y\,A,x\rangle \mid x\in\mathbb{R}^n_+\right\}=\begin{cases}0\,, & \text{if } c+y\,A\geq\mathbb{O},\\ -\infty\,, & \text{otherwise.}\end{cases}$$

Consequently problem (2) has the form

$$\begin{aligned}&\varphi(y):=-\langle b,y\rangle\to\sup\,,\\ &c+y\,A\geq\mathbb{O},\quad y\geq\mathbb{O}\,.\end{aligned}\tag{11}$$

Problems (10) and (11) are equivalent. Their solution sets U^* and Y^* satisfy the conditions $Y^*=-U^*$, and $\varphi^*=\psi^*$.

We have obtained that $f^*=\psi^*=\varphi^*$ and that problem (11) has solution. From theorem 2, under the hypothesis (4), the problem (9) satisfies the global regularity condition.

THEOREM 7. *The solution set of problem* (10) *coincides with the subdifferential of the sensitivity function of problem* (9) *evaluated at point zero, that is* $U^*=\partial F(\mathbb{O})$.

P r o o f. We write the parametric auxiliary problem

$$\begin{aligned}f(x):=\langle c,x\rangle&\to\inf\,,\\ A\,x&\leq b+v\,,\\ P&=\mathbb{R}^n_+\,.\end{aligned}$$

We consider any $u^*\in U^*$ and we show

$$F(v)-F(\mathbb{O})\geq\langle u^*,v\rangle\quad\forall\,v\in\mathbb{R}^s.\tag{12}$$

In fact for all $x\in X(v)$ we have

$$\langle c,x\rangle\geq\langle c,x\rangle+\langle u^*,b+v-A\,x\rangle=\langle b,u^*\rangle+\langle u^*,v\rangle+$$
$$+\langle c-u^*A,x\rangle\geq\langle b,u^*\rangle+\langle u^*,v\rangle=f^*+\langle u^*,v\rangle\,.$$

Then it follows for each $v\in\mathbb{R}^s$

$$F(v)\geq f^*+\langle u^*,v\rangle\,.$$

Taking into account that it is $f^*=F(\mathbb{O})$, (12) follows.

On the other hand let $u^*\in\partial F(\mathbb{O})$, so that (12) is satisfied. By substituting into (12) v by the unit vectors e_i and taking into account that it is $F(e_i)\leq F(\mathbb{O})$, we have $u^*_i\leq 0$. Thus it is $u^*\leq\mathbb{O}$.

Now, from (12) we have

$$F(v) + \langle -u^*, v\rangle \geq F(\mathbb{O}) \qquad \forall\ v \in \mathbb{R}^s.$$

From the lemma at section 1°, $\varphi(-u^*) \geq F(\mathbb{O})$. The relations

$$\varphi^* \geq \varphi(-u^*) \geq F(\mathbb{O}) = f^* \geq \varphi^*$$

ensure that the vector $-u^*$ solves problem (11). In this case, as previously observed, u^* solves (10) and the proof is complete. □

The following result (see [4]) is related to previous theorem.

THEOREM 8. *If u is a feasible solution of the dual problem* (10)*, then $u \in \partial_\varepsilon F(\mathbb{O})$, where $\varepsilon = f^* - \langle b, u\rangle$.*

Proof. We must prove that

$$F(v) - F(\mathbb{O}) \geq \langle u, v\rangle - \varepsilon \qquad \forall\ v \in \mathbb{R}^s.$$

The property is obvious for $X(v) = \varnothing$. Now let $X(v) \neq \varnothing$. For each $x \in X(v)$ we have

$$\langle c, x\rangle \geq \langle c, x\rangle + \langle u, b + v - A\,x\rangle = \langle b, u\rangle + \langle u, v\rangle +$$
$$+ \langle c - uA, x\rangle \geq f^* - \varepsilon + \langle u, v\rangle = F(\mathbb{O}) + \langle u, v\rangle - \varepsilon.$$

The property follows by considering that the above relation holds for all $x \in X(v)$. □

8°. A recent survey on the sensitivity theory in optimization is in [5].

REFERENCES

1. Lazarev A. V. *On duality relationship in Mathematical Programming*, Seminar "DHA & CAGD", Selected Reports, May 17, 2008, (in Russian). (`http://dha.spb.ru/reps08.shtml#0517`).

2. Lazarev A. V. *Necessary conditions for global optimality*, Seminar "DHA & CAGD", Selected Reports, September 9, 2008, (in Russian). (`http://dha.spb.ru/reps08.shtml#0909`).

3. Shapiro J. *Mathematical Programming: Structures and Algorithms.*, J. Wiley & Sons, 1979.

4. De Leone R., Gaudioso M., Monaco M. F. *Nonsmooth optimization methods for parallel decomposition of multicommodity flow problems*, Annals of Operation Research, 1993, Vol. 44, pp. 299–311.

5. Izmailov A. F. *Sensitivity Analysis*, Fizmatlit, Moscow, 2006, (in Russian).