THE BEST LINEAR SEPARATION OF TWO SETS*

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Consider the problem of the best approximate separation of two finite sets in the linear case. This problem is reduced to the problem of nonsmooth optimization, analyzing which we use all power of the linear programming theory.

Ideologically we follow [1].

1°. Suppose we have two finite sets in $\mathbb{R}^n$

$$A = \{a_i\}_{i=1}^m \quad \text{and} \quad B = \{b_j\}_{j=1}^k.$$

The sets A and B are called *strictly separable*, if there exist a nonzero vector $w \in \mathbb{R}^n$ and a real number γ, such that

$$\langle w, a_i \rangle < \gamma \quad \text{for all } i \in 1 : m, \tag{1}$$

$$\langle w, b_j \rangle > \gamma \quad \text{for all } j \in 1 : k. \tag{2}$$

If conditions (1) and (2) are satisfied, it is also said that the hyperplane H defined by the equation $\langle w, x \rangle = \gamma$ *strictly separates* the set A from the set B.

2°. We introduce the function

$$f(g) = \frac{1}{m} \sum_{i=1}^m \big[\langle w, a_i \rangle - \gamma + c\big]_+ + \frac{1}{k} \sum_{j=1}^k \big[-\langle w, b_j \rangle + \gamma + c\big]_+, \tag{3}$$

where $g = (w, \gamma)$, $c > 0$ is a parameter, and $[u]_+ = \max\{0, u\}$. It is clear that $f(g) \geq 0$ for all g.

THEOREM 1. *The sets A and B are strictly separable if and only if there exists a vector g_* such that $f(g_*) = 0$.*

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P r o o f. Let $f(g_*) = 0$ for some vector $g_* = (w_*, \gamma_*)$. First, we show that $w_* \neq \mathbb{O}$. Otherwise

$$f(g_*) = (-\gamma_* + c)_+ + (\gamma_* + c)_+ = \begin{cases} -\gamma_* + c & \text{at } \gamma_* \leq -c, \\ 2c & \text{at } \gamma_* \in [-c, c], \\ \gamma_* + c & \text{at } \gamma_* \geq c. \end{cases}$$

Hence it follows that $f(g_*) \geq 2c$. It contradicts the condition $f(g_*) = 0$.

Furthermore, the condition $f(g_*) = 0$ guarantees that all the terms

$$\big[\langle w_*, a_i \rangle - \gamma_* + c\big]_+ \quad \text{and} \quad \big[-\langle w_*, b_j \rangle + \gamma_* + c\big]_+$$

equal to zero. This is possible only when

$$\langle w_*, a_i \rangle - \gamma_* + c \leq 0 \quad \text{for all } i \in 1:m, \tag{4}$$

$$-\langle w_*, b_j \rangle + \gamma_* + c \leq 0 \quad \text{for all } j \in 1:k. \tag{5}$$

It remains to note that (4) and (5) provide that the conditions of strict separation (1) and (2) are satisfied with $w = w_*$ and $\gamma = \gamma_*$.

Let us prove the converse. Let the conditions (1) and (2) are satisfied. Denote

$$d := \min_{j \in 1:k} \langle w, b_j \rangle - \max_{i \in 1:m} \langle w, a_i \rangle > 0, \tag{6}$$

$$w_* = \left(\tfrac{2c}{d}\right) w, \quad \gamma_* = \tfrac{1}{2}\big[\min_{j \in 1:k} \langle w_*, b_j \rangle + \max_{i \in 1:m} \langle w_*, a_i \rangle\big].$$

According to (6) and the definition of w_*

$$\min_{j \in 1:k} \langle w_*, b_j \rangle - \max_{i \in 1:m} \langle w_*, a_i \rangle = 2c.$$

We have

$$\max_{i \in 1:m} \langle w_*, a_i \rangle = 2\gamma_* - \min_{j \in 1:k} \langle w_*, b_j \rangle = 2\gamma_* - 2c - \max_{i \in 1:m} \langle w_*, a_i \rangle,$$

so

$$\max_{i \in 1:m} \langle w_*, a_i \rangle = \gamma_* - c. \tag{7}$$

Similarly

$$\min_{j \in 1:k} \langle w_*, b_j \rangle = 2\gamma_* - \max_{i \in 1:m} \langle w_*, a_i \rangle = 2\gamma_* + 2c - \min_{j \in 1:k} \langle w_*, b_j \rangle,$$

so

$$\min_{j \in 1:k} \langle w_*, b_j \rangle = \gamma_* + c. \tag{8}$$

Let $g_* = (w_*, \gamma_*)$. By (7) and (8) we get $f(g_*) = 0$.

The theorem is proved. $\square$

3°. In the proof of the theorem 1 we described a transformation of the vector $g = (w, \gamma)$ that generated a strictly separating hyperplane $H = \{x \mid \langle w, x\rangle = \gamma\}$ into the vector $g_* = (w_*, \gamma_*)$ with the property $f(g_*) = 0$. The point is, given the vector g, the value $f(g)$ can be positive (it depends on the parameter c).

EXAMPLE 1. Consider two sets A and B on the plane $\mathbb{R}^2$, each containing a single point $a = (0, 0)$ and $b = (0, 2)$ respectively. The vector $g = (w, \gamma)$ with components $w = (0, 1)$ and $\gamma \in (0, 2)$ generates a line $x_2 = \gamma$ that strictly separates the point a from the point b (see figure 1).

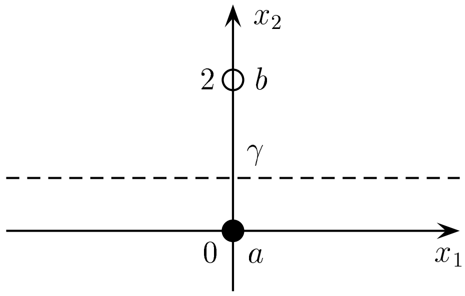

Figure 1

At the same time

$$f(g) = [-\gamma + c]_+ + [-2 + \gamma + c]_+.$$

The figure 2 shows a plot of $f(g)$ as a function of γ for $c \in (0, 1]$.

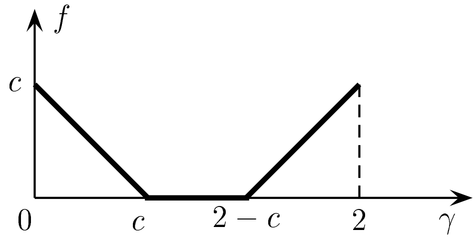

Figure 2

We see that $f(g) = 0$ for $\gamma \in [c, 2 - c]$. At $\gamma \in (0, c) \cup (2 - c, 2)$ the line $x_2 = \gamma$ still strictly separates the point a from the point b, but $f(g) > 0$.

4°. Consider an extremal problem

$$f(g) \to \min, \tag{9}$$

where $f(g)$ is the function of the form (3). This problem is equivalent to a linear programming problem

$$\frac{1}{m}\sum_{i=1}^{m} y_i + \frac{1}{k}\sum_{j=1}^{k} z_j \to \min, \tag{10}$$

$$\begin{gathered} -\langle w, a_i\rangle + \gamma + y_i \geq c, \quad i \in 1:m; \\ \langle w, b_j\rangle - \gamma + z_j \geq c, \quad j \in 1:k; \\ y_i \geq 0,\ i \in 1:m; \quad z_j \geq 0,\ j \in 1:k. \end{gathered}$$

The set of plans of the problem (10) is nonempty (a vector with components $w = \mathbb{O}$, $\gamma = 0$, $y_i \equiv c$, $z_j \equiv c$ is a plan) and the objective function is bounded below by zero. So the problem (10) has a solution. By the equivalence, the problem (9) has a solution too, and the minimum values of the objective functions of these problems are equal. We denote this common value by μ. We also note that if $\big(w_*, \gamma_*, \{u_i^*\}, \{v_j^*\}\big)$ is the solution of (10) then $g_* = \{w_*, \gamma_*\}$ is the solution of (9).

5°. When $\mu = 0$ we get $f(g_*) = 0$. By the theorem 1 the vector $g_* = (w_*, \gamma_*)$ generates a hyperplane $H = \big\{x \mid \langle w_*, x\rangle = \gamma_*\big\}$ that strictly separates the set A from the set B.

The vector g_* can be improved by using the non-uniqueness of the solution of the problem (9). Let

$$\begin{gathered} w_0 = w_*/\|w_*\|, \\ \gamma_0 = \tfrac{1}{2}\big[\min_{j\in 1:k}\langle w_0, b_j\rangle + \max_{i \in 1:m}\langle w_0, a_i\rangle\big], \\ c_0 = \tfrac{1}{2}\big[\min_{j\in 1:k}\langle w_0, b_j\rangle - \max_{i \in 1:m}\langle w_0, a_i\rangle\big], \\ g_0 = (w_0, \gamma_0). \end{gathered}$$

Then for all $i \in 1:m$

$$\langle w_0, a_i\rangle - \gamma_0 + c_0 = \langle w_0, a_i\rangle - \max_{i\in 1:m}\langle w_0, a_i\rangle \leq 0,$$

and for all $j \in 1:k$

$$-\langle w_0, b_j\rangle + \gamma_0 + c_0 = -\langle w_0, b_j\rangle + \min_{j\in 1:k}\langle w_0, b_j\rangle \leq 0.$$

This means that $f(g_0) = 0$ for $c = c_0$. The hyperplane $H_0 = \big\{x \mid \langle w_0, x\rangle = \gamma_0\big\}$ strictly separates the set A from the set B, and the width of the dividing strip is equal to $2c_0$.

6°. As noted in section 4°, the problem (9) always has a solution. When $\mu > 0$, according to the theorem 1, the sets A and B can not be strictly separated. In this case we say that the hyperplane $H_* = \{x \mid \langle w_*, x\rangle = \gamma_*\}$ generated by the solution $g_* = (w_*, \gamma_*)$ of the problem (9) is *the best hyperplane approximately separating the set A from the set B* (for a given value of the parameter c).

However, there is a catch: there is no guarantee that the component w_* of the vector g_* is nonzero. Let us examine this situation.

THEOREM 2. *The problem* (9) *has a solution $g_* = (w_*, \gamma_*)$ with $w_* = \mathbb{O}$ if and only if the following condition holds:*

$$\frac{1}{m}\sum_{i=1}^{m} a_i = \frac{1}{k}\sum_{j=1}^{k} b_j. \tag{11}$$

Proof. Necessity. When $w_* = \mathbb{O}$, it is easy to calculate the extreme value of the objective function of the linear programming problem (10). Indeed,

$$\mu = f(g_*) = \min_{\gamma}\big\{[-\gamma + c]_+ + [\gamma + c]_+\big\} = 2c.$$

Of the same extreme value is the linear programming problem that is dual to (10). By the solvability of the dual problem, the following system is consistent:

$$c\left(\sum_{i=1}^{m} u_i + \sum_{j=1}^{k} v_j\right) = 2c, \tag{12}$$

$$-\sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{k} v_j b_j = \mathbb{O}, \tag{13}$$

$$\sum_{i=1}^{m} u_i - \sum_{j=1}^{k} v_j = 0, \tag{14}$$

$$0 \le u_i \le \tfrac{1}{m},\ i \in 1:m; \quad 0 \le v_j \le \tfrac{1}{k},\ j \in 1:k. \tag{15}$$

From (12) and (14) it follows that

$$\sum_{i=1}^{m} u_i = 1, \quad \sum_{j=1}^{k} v_j = 1.$$

Taking into account (15), we conclude that all u_i are equal to $\frac{1}{m}$ and all v_j are equal to $\frac{1}{k}$. Now (13) is equivalent to (11).

Sufficiency. We write the problem dual to (10):

$$c\left(\sum_{i=1}^{m} u_i + \sum_{j=1}^{k} v_j\right) \to \max$$

subject to constraints (13)–(15). By (11) the set of $u_i \equiv \frac{1}{m}$, $v_j \equiv \frac{1}{k}$ is a plan of this problem. The objective function value is equal to $2c$.

At the same time, the set

$$w = \mathbb{O}, \quad \gamma = 0, \quad y_i \equiv c, \quad z_j \equiv c \tag{16}$$

is a plan of the problem (10) and the objective function value is also equal to $2c$. Hence it follows that the plan (16) of the problem (10) with $w = \mathbb{O}$ is optimal.

The theorem is proved. $\square$

EXAMPLE 2. Consider two sets on the plane $\mathbb{R}^2$:

$$A = \{(0,0),(1,1)\}, \quad B = \{(1,0),(0,1)\}$$

(see figure 3). In this case the condition (11) holds. By the theorem 2, the problem (9) has a solution $g_* = (w_*, \gamma_*)$ with $w_* = \mathbb{O}$. In this case $\mu = 2c$.

We will show that the problem (9) has another solution $g_0 = (w_0, \gamma_0)$ with $w_0 \neq \mathbb{O}$.

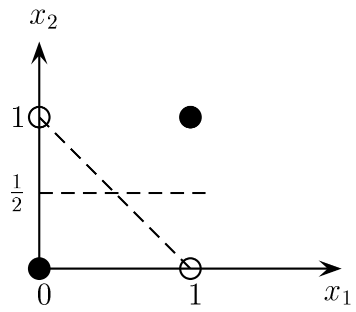

Figure 3

By (3)

$$f(g) = \tfrac{1}{2}\{[-\gamma + c]_+ + [w^1 + w^2 - \gamma + c]_+\} + \tfrac{1}{2}\{[-w^1 + \gamma + c]_+ + [-w^2 + \gamma + c]_+\}.$$

Here $w = (w^1, w^2)$. Let

$$w_0 = (c, c), \quad \gamma_0 = c, \quad g_0 = (w_0, \gamma_0).$$

Then $f(g_0) = 2c$. So, a minimum of the function $f(g)$ is attained on the vector g_0. The hyperplane $H_0 = \{x \mid x_1 + x_2 = 1\}$ is the best hyperplane approximately separating the set A from the set B.

Of the same property are the vector $g_1 = (w_1, \gamma_1)$ with $w_1 = (0, c)$ and $\gamma_1 = \frac{c}{2}$ and the hyperplane $H_1 = \{x \mid x_2 = \frac{1}{2}\}$ (see figure 3).

7°. The peculiarity noted in example 2 has a general nature.

THEOREM 3. *When $\mu > 0$, the problem (9) has a solution $g_0 = (w_0, \gamma_0)$ with $w_0 \neq \mathbb{O}$.*

P r o o f. Assume that the solution $g_* = (w_*, \gamma_*)$ of the problem (9) has zero component w_*. We will construct another solution $g_0 = (w_0, \gamma_0)$ with $w_0 \neq \mathbb{O}$.

By the theorem 2 the relation (11) holds and $\mu = 2c$. Take any nonzero vector $h \in \mathbb{R}^n$ and consider a linear programming problem

$$
\begin{gathered}
\langle h, w \rangle \to \min, \qquad (17) \\
-\frac{1}{m}\sum_{i=1}^{m} y_i - \frac{1}{k}\sum_{j=1}^{k} z_j = -2c; \\
-\langle w, a_i \rangle + \gamma + y_i \geq c, \quad i \in 1:m; \\
\langle w, b_j \rangle - \gamma + z_j \geq c, \quad j \in 1:k; \\
y_i \geq 0,\ i \in 1:m; \quad z_j \geq 0,\ j \in 1:k.
\end{gathered}
$$

The vector (16) satisfies the constrains of the problem (17), so it is its plan. We will show that this plan can not be optimal.

Indeed, if the plan (16) is optimal then the problem dual to (17) must have a plan with the same (i. e. zero) value of the objective function. Thus, the following system must be consistent:

$$
c\left(\sum_{i=1}^{m} u_i + \sum_{j=1}^{k} v_j - 2\zeta\right) = 0, \qquad (18)
$$

$$
-\sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{k} v_j b_j = h, \qquad (19)
$$

$$
\sum_{i=1}^{m} u_i - \sum_{j=1}^{k} v_j = 0, \qquad (20)
$$

$$
0 \leq u_i \leq \tfrac{1}{m}\zeta,\ i \in 1:m; \quad 0 \leq v_j \leq \tfrac{1}{k}\zeta,\ j \in 1:k. \qquad (21)
$$

However, it can be shown that this system is inconsistent.

From (18) and (20) it follows that

$$
\sum_{i=1}^{m} u_i = \zeta, \quad \sum_{j=1}^{k} v_j = \zeta.
$$

By (21) we obtain $u_i \equiv \frac{1}{m}\zeta$, $v_j \equiv \frac{1}{k}\zeta$. Equality (19) takes the form

$$\zeta\left(-\frac{1}{m}\sum_{i=1}^{m} a_i + \frac{1}{k}\sum_{j=1}^{k} b_j\right) = h.$$

But this contradicts to (11) (recall that $h \neq \mathbb{O}$).

It is ascertained that the plan (16) of the problem (17) with a zero value of the objective function is not optimal. Hence, there exist a plan

$$\left(w_0,\ \gamma_0,\ \{u_i^0\},\ \{v_j^0\}\right) \tag{22}$$

with a negative value of the objective function. Such a plan must be with $w_0 \neq \mathbb{O}$.

Now we note that the plan (22) of the problem (17) satisfies the constraints of (10) and on it the objective function of the problem (10) takes the smallest possible value equal to $2c$ (recall that $\mu = 2c$). By the equivalence of the problems (9) and (10) the vector $g_0 = (w_0, \gamma_0)$ with $w_0 \neq \mathbb{O}$ is a solution of the problem (9).

The theorem is proved. □

Remark. As a non-zero vector h we can take, for example, any non-zero difference $b_{j_0} - a_{i_0}$. In this case, a set of plans of the problem dual to the problem (17), which is defined by (19)– (21), will not be empty. Together with the nonempty set of plans of the problem (17) this guarantees the existence of the optimal plan of the problem (17).

8°. When $\mu > 0$ the solution $g_0 = (w_0, \gamma_0)$ of the problem (9) with $w_0 \neq \mathbb{O}$ can be reduced to the canonical form. As in section 5° we set

$$\begin{aligned} w_1 &= w_0/\|w_0\|, \\ \gamma_1 &= \tfrac{1}{2}\big[\min_{j\in 1:k}\langle w_1, b_j\rangle + \max_{i\in 1:m}\langle w_1, a_i\rangle\big], \\ c_1 &= \tfrac{1}{2}\big[\min_{j\in 1:k}\langle w_1, b_j\rangle - \max_{i\in 1:m}\langle w_1, a_i\rangle\big], \\ g_1 &= (w_1, \gamma_1). \end{aligned}$$

In this case $c_1 \leq 0$. When $c_1 = 0$ the hyperplane $H_1 = \big\{x \mid \langle w_1, x\rangle = \gamma_1\big\}$ non-strictly separates the set A from the set B. When $c_1 < 0$ the same hyperplane H_1 is the best approximately separating the set A from the set B.

By definition of w_1, γ_1, c_1 we have

$$\begin{aligned} \langle w_1, a_i\rangle - \gamma_1 + c_1 \leq 0, &\quad i \in 1:m \\ -\langle w_1, b_j\rangle + \gamma_1 + c_1 \leq 0, &\quad j \in 1:k. \end{aligned}$$

When $c_1 < 0$ these inequalities define a “mixed strip”

$$c_1 \leq \langle w_1, x\rangle - \gamma_1 \leq -c_1,$$

which contains both the points of the set A and the points of the set B. The width of the mixed strip is equal to $2|c_1|$.

9°. The example of the best approximate separation of two sets is illustrated on the figure 4.

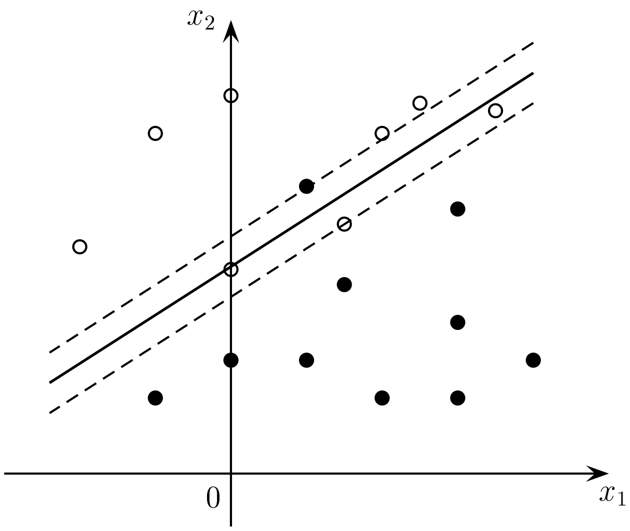

Figure 4

REFERENCES

1. Bennett K. P., Mangassarian O. L. *Robust linear programming discrimination of two linearly inseparable sets* // Optimization Methods and Software. 1992. Vol. 1. P. 23–34.