SYMMETRY OF PONS MATRICES*

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1°. PONS matrices are introduced in [1]. It is a recurrent sequence of square matrices P_1, P_2, ..., each of those is of a twice bigger order than the previous one. The sequence starts with Hadamard matrix

$$P_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

To get from the matrix P_{n-1} to P_n the rows of the initial matrix are paired up, and each *two* rows of the matrix P_{n-1} generate *four* rows of the matrix P_n of a twice bigger length according to the following rule:

$$\begin{pmatrix} \dots \\ A \\ B \\ \dots \end{pmatrix} \longrightarrow \begin{pmatrix} \dots & \dots \\ A & B \\ A & -B \\ B & A \\ -B & A \\ \dots & \dots \end{pmatrix} \tag{1}$$

In particular,

$$P_2 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{pmatrix}.$$

In report [2], the following theorem is stated as a hypothesis.

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THEOREM 1. *The matrices P_n for all $n = 1, 2, \dots$ are symmetric.*

Symmetry of the matrices P_1 and P_2 is evident. It's not difficult to verify symmetry of the matrix P_3 as well. Proving the theorem in a general case requires some preparations.

2°. Square matrix P_n has an order of 2^n. We represent the indices $k,\ j \in 0 : 2^n - 1$ of a matrix element $P_n[k, j]$ in a form

$$k = 4m + m', \quad m \in 0 : 2^{n-2} - 1,\ m' \in 0 : 3;$$

$$j = l + 2^{n-1} l', \quad l \in 0 : 2^{n-1} - 1,\ l' \in 0 : 1.$$

Then the relation (1) can be represented as a set of equalities

$$\begin{aligned}
P_n[4m, l] &= P_{n-1}[2m, l], \\
P_n[4m, l + 2^{n-1}] &= P_{n-1}[2m + 1, l], \\
P_n[4m + 1, l] &= P_{n-1}[2m, l], \\
P_n[4m + 1, l + 2^{n-1}] &= -P_{n-1}[2m + 1, l], \\
P_n[4m + 2, l] &= P_{n-1}[2m + 1, l], \\
P_n[4m + 2, l + 2^{n-1}] &= P_{n-1}[2m, l], \\
P_n[4m + 3, l] &= P_{n-1}[2m + 1, l], \\
P_n[4m + 3, l + 2^{n-1}] &= -P_{n-1}[2m, l], \\
m \in 0 : 2^{n-2} - 1, &\quad l \in 0 : 2^{n-1} - 1.
\end{aligned} \tag{2}$$

We can compactly write formulas (2) in a single line if we use the representation of the indices in a binary code

$$k = (k_{n-1}, \dots, k_0)_2, \quad j = (j_{n-1}, \dots, j_0)_2, \quad k_\nu, j_\nu \in 0 : 1.$$

We have

$$\begin{aligned}
&P_n[(k_{n-1}, \dots, k_2, k_1, k_0)_2, (j_{n-1}, j_{n-2}, \dots, j_0)_2] = \\
&\qquad = (-1)^{k_0(k_1 + j_{n-1})} P_{n-1}[(k_{n-1}, \dots, k_2, k_1 \oplus j_{n-1})_2, (j_{n-2}, \dots, j_0)_2],
\end{aligned} \tag{3}$$

where $\oplus$ is a modulo 2 addition.

3°. Using binary codes also helps to write an explicit formula for the elements of the matrix P_n. It is obvious that

$$P_1[k,j] = P_1[(k_0)_2, (j_0)_2] = (-1)^{k_0 j_0}. \tag{4}$$

It is a bit less obvious that

$$P_2[k,j] = P_1[(k_1,k_0)_2, (j_1,j_0)_2] = (-1)^{(k_0+j_0)(k_1+j_1)}. \tag{5}$$

How does the formula for the element $P_n[k,j]$ in a general case look like? The answer to this question is given by the following lemma.

LEMMA 1. *For each $n = 1, 2, \ldots$ it holds*

$$P_n[k,j] = \prod_{\nu=0}^{n-1} (-1)^{(k_\nu + j_{n-\nu})(k_{\nu+1} + j_{n-1-\nu})}, \tag{6}$$

where for the sake of convenience we put $k_n = j_n := 0$.

P r o o f. We will prove the equality (6) by induction on n. When $n = 1$ and $n = 2$, the equality (6) coincides with (4) and (5), respectively. Let's make an induction step from $n-1$ to n for $n \geq 3$.

We fix the indices $k, j \in 0 : 2^n - 1$, $k = (k_{n-1}, \ldots, k_0)_2$, $j = (j_{n-1}, \ldots, j_0)_2$. We put

$$k'_0 := k_1 \oplus j_{n-1},$$

$$k'_\nu := k_{\nu+1}, \quad \nu \in 1 : n-1;$$

$$j'_\nu := j_\nu, \quad \nu \in 0 : n-2,$$

$$j'_{n-1} := 0.$$

Taking into account these notations and the induction hypothesis we can rewrite the right side of the equality (3) in a form

$$(-1)^{k_0(k_1+j_{n-1})} P_{n-1}[(k'_{n-2}, \ldots, k'_1, k'_0)_2, (j'_{n-2}, \ldots, j'_0)_2] =$$

$$= (-1)^{k_0(k_1+j_{n-1})} \prod_{\nu=0}^{n-2} (-1)^{(k'_\nu + j'_{n-1-\nu})(k'_{\nu+1} + j'_{n-2-\nu})} =$$

$$= (-1)^{k_0(k_1+j_{n-1})} (-1)^{(k_1 \oplus j_{n-1})(k_2+j_{n-2})} \prod_{\nu=1}^{n-2} (-1)^{(k_{\nu+1} + j_{n-1-\nu})(k_{\nu+2} + j_{n-2-\nu})}.$$

Substituting this into (3) and noticing that $(-1)^{a \oplus b} = (-1)^{a+b}$ we get

$$P_n[k,j] = (-1)^{(k_0+j_n)(k_1+j_{n-1})} (-1)^{(k_1+j_{n-1})(k_2+j_{n-2})} \prod_{\nu'=2}^{n-1} (-1)^{(k_{\nu'} + j_{n-\nu'})(k_{\nu'+1} + j_{n-1-\nu'})},$$

which is equivalent to (6).

The lemma is proved. □

4°. Now we address the proof of the theorem. Essentially, it is required to verify that for each $n = 1, 2, \dots$ the following identities hold:

$$P_n[k, j] = P_n[j, k], \quad k, j \in 0 : 2^n - 1.$$

Formula (6) helps to verify it immediately. We have

$$P_n[j, k] = \prod_{\nu=0}^{n-1} (-1)^{(j_\nu + k_{n-\nu})(j_{\nu+1} + k_{n-1-\nu})} =$$

$$= \prod_{\mu=0}^{n-1} (-1)^{(j_{n-1-\mu} + k_{\mu+1})(j_{n-\mu} + k_\mu)} = P_n[k, j].$$

The theorem is proved. □

5°. Let $N = 2^n$. Denote the k-th row of the matrix P_n by a symbol p_k and consider it as a signal from a space $\mathbb{C}_N$ defined on the main period:

$$p_k(j) = P_n[k, j], \quad j \in 0 : N - 1, \quad k = 0, 1, \dots, N - 1.$$

There is another hypothesis stated in [2] that we present here as a theorem.

THEOREM 2. *For all $n \geq 2$ and $k, k' \in 0 : \frac{N}{2} - 1$, the following equality holds:*

$$p_k(j) p_{k'}(j) = \alpha^n_{k,k'}\, p_{\frac{N}{2}+k}(j) p_{\frac{N}{2}+k'}(j), \quad j \in 0 : N - 1, \tag{7}$$

where

$$\alpha^n_{k,k'} = (-1)^{k_{n-2} + k'_{n-2}}.$$

For $n = 2$ formula (7) takes a form

$$p_k(j) p_{k'}(j) = (-1)^{k+k'}\, p_{k+1}(j) p_{k'+1}(j), \quad j \in 0 : 3, \quad k, k' \in 0 : 1.$$

The validity of this identity can be verified directly.

The proof of the theorem for $n \geq 3$ derives from the following auxiliary statement.

LEMMA 2. *For all $n \geq 3$ and $k \in 0 : \frac{N}{2} - 1$, the following equality holds:*

$$p_k(j) = (-1)^{(k_{n-2} + j_2 + j_0)} p_{\frac{N}{2}+k}(j), \quad j \in 0 : N - 1. \tag{8}$$

P r o o f. We rewrite formula (6) detaching the last two factors:

$$P_n[k,j] = (-1)^{(k_{n-1}+j_1)j_0}(-1)^{(k_{n-2}+j_2)(k_{n-1}+j_1)}\prod_{\nu=0}^{n-3}(-1)^{(k_\nu+j_{n-\nu})(k_{\nu+1}+j_{n-1-\nu})} =$$
$$= (-1)^{(k_{n-1}+j_1)(k_{n-2}+j_2+j_0)}\varphi_k^n(j), \tag{9}$$

where for the sake of brevity we define

$$\varphi_k^n(j) = \prod_{\nu=0}^{n-3}(-1)^{(k_\nu+j_{n-\nu})(k_{\nu+1}+j_{n-1-\nu})}.$$

Fix $k \in 0 : \frac{N}{2}-1$, $k = (0, k_{n-2}, \ldots, k_0)_2$. Then the binary code of the integer $\frac{N}{2}+k$ will be $(1, k_{n-2}, \ldots, k_0)_2$. In accordance with (9) we have

$$p_k(j) = (-1)^{j_1(k_{n-2}+j_2+j_0)}\varphi_k^n(j),$$
$$p_{\frac{N}{2}+k}(j) = (-1)^{(j_1+1)(k_{n-2}+j_2+j_0)}\varphi^n_{\frac{N}{2}+k}(j).$$

It is remaining to note that the definition of φ_k^n includes only the least significant bits $k_0, k_1, \ldots, k_{n-2}$ of the number k, thus $\varphi^n_{\frac{N}{2}+k}(j) \equiv \varphi_k^n(j)$. The validity of equality (8) is ascertained. □

6°. A consequence of the theorem 1 is the fact that PONS matrices can be built by columns rather than by rows. By analogy with the scheme (1), to get from the matrix P_{n-1} to P_n the columns of the initial matrix are paired up and each two columns of the matrix P_{n-1} generate four columns of the matrix P_n of a twice bigger height by the following rule:

$$\begin{pmatrix} \vdots & A & B & \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots & A & A & B & -B & \vdots \\ \vdots & B & -B & A & A & \vdots \end{pmatrix} \tag{10}$$

The scheme (10) seems to be more promising from the point of proving the rest two hypotheses stated in [2].

REFERENCES

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