SHAPIRO POLYNOMIALS OF THE SECOND KIND*

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1°. The Shapiro polynomials are defined by the following recursive relations

$$\begin{aligned} P_{m+1}(x) &= P_m(x) + x^{2^m} Q_m(x), \\ Q_{m+1}(x) &= P_m(x) - x^{2^m} Q_m(x). \end{aligned} \tag{1}$$

for $m = 0, 1, \ldots$, and the initial conditions

$$P_0(x) \equiv 1, \quad Q_0(x) \equiv 1$$

(see [1, 2]). The polynomial $Q_m(x)$ is called *complementary* to the polynomial $P_m(x)$. Both the polynomials $P_m(x)$ and $Q_m(x)$ are of degree $2^m - 1$.

The coefficients of the Shapiro polynomials take values ± 1. We can write explicit formulas for them. Let

$$P_m(x) = \sum_{k=0}^{2^m-1} a_k\, x^k, \quad Q_m(x) = \sum_{k=0}^{2^m-1} b_k\, x^k.$$

We relate to each index $k \in 0 : 2^m - 1$ its binary expansion

$$k = (k_{m-1}, k_{m-2}, \ldots, k_0)_2,$$

where $k_\alpha \in \{0, 1\}$. Then, (see [2–4]) for $m \geq 2$ and $k \in 0 : 2^m - 1$

$$a_k = (-1)^{\sum_{\alpha=1}^{m-1} k_{\alpha-1}\, k_\alpha}, \quad b_k = (-1)^{\sum_{\alpha=1}^{m-1} k_{\alpha-1}\, k_\alpha + k_{m-1}}. \tag{2}$$

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The value
$$\varphi(k) = \sum_{\alpha=1}^{m-1} k_{\alpha-1}\, k_\alpha$$
shows, how many times the block $[1 \quad 1]$ appears in the binary expansion of the index k.

We define the Shapiro polynomials of the second kind by the same recursive relations (1), but with different initial conditions
$$P_0(x) \equiv 1, \quad Q_0(x) \equiv -1.$$
We denote the corresponding polynomials by $\tilde{P}_m(x)$ and $\tilde{Q}_m(x)$. In this work we investigate their properties, including their relation to the Shapiro polynomials of the first kind.

2°. From (1), in particular, it follows that
$$\begin{aligned}
&\tilde{P}_1(x) = 1 - x, && \tilde{Q}_1(x) = 1 + x,\\
&\tilde{P}_2(x) = 1 - x + x^2 + x^3, && \tilde{Q}_2(x) = 1 - x - x^2 - x^3,\\
&\tilde{P}_3(x) = 1 - x + x^2 + x^3 + && \tilde{Q}_3(x) = 1 - x + x^2 + x^3 -\\
&\qquad + x^4 - x^5 - x^6 - x^7, && \qquad - x^4 + x^5 + x^6 + x^7.
\end{aligned}$$

LEMMA 1. *The following formulas hold*
$$\tilde{P}_{m+1}(x) = \tilde{P}_m(-x^2) - x\,\tilde{P}_m(x^2), \quad m = 0, 1, \ldots; \tag{3}$$
$$\tilde{Q}_{m+1}(x) = \tilde{Q}_m(-x^2) - x\,\tilde{Q}_m(x^2), \quad m = 1, 2, \ldots. \tag{4}$$

Proof. The identity (3) for $m = 0, 1, 2$ and the identity (4) for $m = 1, 2$ are verified directly. We make the induction step from m to $m+1$, assuming that $m \geq 2$.

We rewrite the relations (1), replacing in them m by $m-1$ and x by x^2:
$$\begin{aligned}
\tilde{P}_m(x^2) &= \tilde{P}_{m-1}(x^2) + x^{2^m}\tilde{Q}_{m-1}(x^2),\\
\tilde{Q}_m(x^2) &= \tilde{P}_{m-1}(x^2) - x^{2^m}\tilde{Q}_{m-1}(x^2).
\end{aligned} \tag{5}$$
Now we replace in (1) m by $m-1$ and x by $-x^2$. Given the equality $(-x^2)^{2^{m-1}} = x^{2^m}$, which is true for $m \geq 2$, we get the relations
$$\begin{aligned}
\tilde{P}_m(-x^2) &= \tilde{P}_{m-1}(-x^2) + x^{2^m}\tilde{Q}_{m-1}(-x^2),\\
\tilde{Q}_m(-x^2) &= \tilde{P}_{m-1}(-x^2) - x^{2^m}\tilde{Q}_{m-1}(-x^2).
\end{aligned} \tag{6}$$

Using the induction hypothesis and the formulas (1), (5), (6) we get the formula (3). Indeed,

$$\begin{aligned}\tilde{P}_{m+1}(x) = \tilde{P}_m(x) + x^{2^m}\tilde{Q}_m(x) = \tilde{P}_{m-1}(-x^2) - x\tilde{P}_{m-1}(x^2) + \\ + x^{2^m}\big[\tilde{Q}_{m-1}(-x^2) - x\,\tilde{Q}_{m-1}(x^2)\big] = \tilde{P}_{m-1}(-x^2) + x^{2^m}\tilde{Q}_{m-1}(-x^2) - \\ - x\big[\tilde{P}_{m-1}(x^2) + x^{2^m}\tilde{Q}_{m-1}(x^2)\big] = \tilde{P}_m(-x^2) - x\,\tilde{P}_m(x^2).\end{aligned}$$

The formula (4) is verified in a similar manner. □

3°. Let

$$\tilde{P}_m(x) = \sum_{k=0}^{2^m-1} c_k\,x^k, \quad \tilde{Q}_m(x) = \sum_{k=0}^{2^m-1} d_k\,x^k.$$

LEMMA 2. *The following recursive relations of the coefficients c_k hold: $c_0 = 1$ and*

$$c_{2k} = (-1)^k c_k, \quad c_{2k+1} = -c_k \tag{7}$$

for $k \in 0 : 2^{m-1} - 1$ and $m = 1, 2, \ldots$.

P r o o f. According to (3)

$$\begin{aligned}\tilde{P}_m(x) &= \tilde{P}_{m-1}(-x^2) - x\tilde{P}_{m-1}(x^2) = \\ &= \sum_{k=0}^{2^{m-1}-1} c_k\,(-x^2)^k - \sum_{k=0}^{2^{m-1}-1} c_k\,x^{2k+1}.\end{aligned}$$

The needed relations obviously follow from this formula. □

The recursive relations (7) for $k \geq 1$ can be combined into a single formula

$$c_{2k+\sigma} = (-1)^{k(\sigma+1)+\sigma} c_k, \tag{8}$$

where $\sigma \in \{0, 1\}$. In addition, $c_0 = 1$, $c_1 = -1$.

4°. We now obtain an explicit representation for the coefficients c_k of the polynomial $\tilde{P}_m(x)$. Let $k = (k_{m-1}, k_{m-2}, \ldots, k_0)_2$ be the binary expansion of the index $k \in 0 : 2^m - 1$.

THEOREM 1. *For $k \in 0 : 2^m - 1$, $m \geq 2$ the following formula holds*

$$c_k = (-1)^{\sum_{\alpha=1}^{m-1} k_{\alpha-1}k_\alpha + k_0}. \tag{9}$$

P r o o f. By induction. For $m = 2$, when $k = (k_1, k_0)_2$, the formula (9) is verified directly (considering that $c_2 = 1$, $c_3 = 1$). We make the induction step from m to $m + 1$, assuming that $m \geq 2$.

Let $k \in 0 : 2^{m+1} - 1$. We represent k in the form $k = 2k' + \sigma$, where $k' \in 0 : 2^m - 1$ and $\sigma \in \{0, 1\}$. Let $k' = (k'_{m-1}, k'_{m-2}, \dots, k'_0)_2$. Then,

$$k = (k'_{m-1}, \dots, k'_0, \sigma)_2.$$

According to (8)

$$c_{2k'+\sigma} = (-1)^{k'(\sigma+1)+\sigma}\, c_{k'} = (-1)^{k'_0(\sigma+1)+\sigma}\, c_{k'}.$$

We use the induction hypothesis,

$$c_{k'} = (-1)^{\sum_{\alpha=1}^{m-1} k'_{\alpha-1}k'_\alpha + k'_0}$$

to obtain

$$\begin{aligned} c_k = c_{2k'+\sigma} &= (-1)^{k'_0(\sigma+1)+\sigma+\sum_{\alpha=1}^{m-1} k'_{\alpha-1}k'_\alpha + k'_0} = \\ &= (-1)^{k'_0\sigma+\sigma+\sum_{\alpha=2}^{m} k_{\alpha-1}k_\alpha} = (-1)^{\sum_{\alpha=1}^{m} k_{\alpha-1}k_\alpha + k_0}. \end{aligned}$$

The theorem is proved. □

According to (1) for $m \geq 1$

$$Q_m(x) = P_{m-1}(x) - [P_m(x) - P_{m-1}(x)].$$

Therefore, for $m \geq 1$

$$\begin{aligned} d_k &= c_k \quad \text{when } k \in 0 : 2^{m-1} - 1, \\ d_k &= -c_k \quad \text{when } k \in 2^{m-1} : 2^m - 1. \end{aligned} \tag{10}$$

Based on (9) and (10) we conclude that for $m \geq 2$

$$d_k = (-1)^{\sum_{\alpha=1}^{m-1} k_{\alpha-1}k_\alpha + k_0 + k_{m-1}}, \quad k \in 0 : 2^m - 1. \tag{11}$$

5°. Denote by **a**, **b**, **c**, **d** the vectors of the coefficients of the Shapiro polynomials of the first and the second kind.

THEOREM 2. *The vectors* **a**, **b**, **c**, **d** *are mutually orthogonal.*

P r o o f. The proof follows from the formulas (2) and (9), (11), if we consider that

$$\sum_{k=0}^{2^m-1} (-1)^{k_0} = 0, \quad \sum_{k=0}^{2^m-1} (-1)^{k_{m-1}} = 0.$$

□

6°. We continue investigating the Shapiro polynomials of the second kind.

LEMMA 3. *For $m \geq 2$ we have the relation*

$$c_{2^m-1-k} = (-1)^{m+k}\, c_k, \quad k \in 0 : 2^{m-1}-1. \tag{12}$$

P r o o f. When $m = 2$ the equality (12) is verified directly. We assume that $m \geq 3$.

Let $k \in 0 : 2^{m-1}-1$, $k = (k_{m-2}, \ldots, k_0)_2$. According to (9),

$$c_k = (-1)^{\sum_{\alpha=1}^{m-2} k_{\alpha-1}\, k_\alpha + k_0}.$$

Denote $k' := 2^m - 1 - k = (1,\, 1-k_{m-2}, \ldots, 1-k_0)_2$. By (9),

$$\begin{aligned} c_{k'} &= (-1)^{\sum_{\alpha=1}^{m-2}(1-k_{\alpha-1})(1-k_\alpha)+1-k_{m-2}+1-k_0} = \\ &= (-1)^{m-2\sum_{\alpha=1}^{m-2} k_\alpha - k_0 + k_{m-2} + \sum_{\alpha=1}^{m-2} k_{\alpha-1}k_\alpha - k_{m-2} - k_0} = (-1)^{m+k} c_k. \end{aligned}$$

The Lemma is proved. □

The formula (12) can be written in the following equivalent form

$$c_{2^m-1-k} = (-1)^{m+k+1}\, c_k, \quad k \in 2^{m-1} : 2^m - 1. \tag{13}$$

Indeed, let $k' = 2^m - 1 - k,\ k' \in 0 : 2^{m-1}-1$. According to (12), we have

$$c_{2^m-1-k'} = (-1)^{m+k'} c_{k'},$$

which is equivalent to (13).

7°. We now consider the Shapiro polynomials of the second kind as functions of the complex variable z. Of particular interest is the case when $|z| = 1$.

THEOREM 3. *For $m \geq 1$ the following identities hold*

$$\tilde{P}_m(-z) = (-1)^{m+1} z^{2^m-1} \tilde{Q}_m(\tfrac{1}{z}), \tag{14}$$

$$\tilde{Q}_m(-z) = (-1)^m z^{2^m-1} \tilde{P}_m(\tfrac{1}{z}). \tag{15}$$

P r o o f. For $m = 1$ these identities can be verified directly. We will assume that $m \geq 2$.

According to (10) we have

$$\begin{aligned} (-1)^{m+1} z^{2^m-1}\, \tilde{Q}_m(\tfrac{1}{z}) &= (-1)^{m+1} \sum_{k=0}^{2^m-1} d_k\, z^{2^m-1-k} = \\ &= (-1)^{m+1} \Bigg[\sum_{k=0}^{2^{m-1}-1} c_k\, z^{2^m-1-k} - \sum_{k=2^{m-1}}^{2^m-1} c_k\, z^{2^m-1-k} \Bigg]. \end{aligned}$$

Along with that, by (13)

$$\sum_{k=0}^{2^{m-1}-1} c_k\, z^{2^m-1-k} = \sum_{k=2^{m-1}}^{2^m-1} c_{2^m-1-k}\, z^k = \sum_{k=2^{m-1}}^{2^m-1} (-1)^{m+k+1} c_k\, z^k$$

and by (12),

$$-\sum_{k=2^{m-1}}^{2^m-1} c_k\, z^{2^m-1-k} = -\sum_{k=0}^{2^{m-1}-1} c_{2^m-1-k}\, z^k = \sum_{k=0}^{2^{m-1}-1} (-1)^{m+k+1} c_k\, z^k.$$

Therefore,

$$(-1)^{m+1} z^{2^m-1}\, \tilde{Q}_m(\tfrac{1}{z}) = \sum_{k=0}^{2^m-1} (-1)^k c_k\, z^k = \tilde{P}_m(-z).$$

The validity of the identity (14) is established.

We substitute in (14) z by $-\frac{1}{z}$ to obtain

$$\tilde{P}_m(\tfrac{1}{z}) = (-1)^{m+1}(-\tfrac{1}{z})^{2^m-1}\tilde{Q}_m(-z) = (-1)^m z^{-2^m+1}\tilde{Q}_m(-z),$$

which is equivalent to (15). The theorem is proved. □

8°. Let $\theta(m)$ be the number of positive coefficients of the Shapiro polynomial of the first kind $P_m(x)$. It is known that (see [2, 4]) for $m \geq 0$

$$\begin{aligned} \theta(2m) &= 2^{2m-1} + 2^{m-1}, \\ \theta(2m+1) &= 2^{2m} + 2^m. \end{aligned} \tag{16}$$

We now obtain the corresponding formulas for the complementary polynomial $Q_m(x)$. Denote the number of its positive coefficients by $\eta(m)$. By definition, $\eta(1) = 1$.

LEMMA 4. *For $m \geq 1$ the following equalities hold*

$$\begin{aligned} \eta(2m) &= 2^{2m-1} + 2^{m-1}, \\ \eta(2m+1) &= 2^{2m}. \end{aligned} \tag{17}$$

P r o o f. From the recursive relations (1) for the Shapiro polynomials follows that, for $m \geq 1$

$$\begin{aligned} \theta(m) &= \theta(m-1) + \eta(m-1), \\ \eta(m) &= \theta(m-1) + [2^{m-1} - \eta(m-1)]. \end{aligned}$$

We add up this equalities to obtain

$$\eta(m) = 2^{m-1} + 2\theta(m-1) - \theta(m). \tag{18}$$

Now replace in (18) m first by $2m$, and then by $2m+1$, and use the formulas (16). We obtain the relations

$$\eta(2m) = 2^{2m-1} + 2[2^{2m-2} + 2^{m-1}] - 2^{2m-1} - 2^{m-1} = 2^{2m-1} + 2^{m-1},$$
$$\eta(2m+1) = 2^{2m} + 2[2^{2m-1} + 2^{m-1}] - 2^{2m} - 2^{m} = 2^{2m}.$$

The lemma is proved. □

Denote by $\tilde{\theta}(m)$ and $\tilde{\eta}(m)$ the number of positive coefficients of the Shapiro polynomials of the second kind $\tilde{P}_m(x)$ and $\tilde{Q}_m(x)$ respectively.

THEOREM 4. *For $m \geq 1$ the following equalities hold*

$$\tilde{\theta}(2m) = \theta(2m), \tag{19}$$
$$\tilde{\theta}(2m+1) = \eta(2m+1), \tag{20}$$
$$\tilde{\eta}(2m) = 2^{2m} - \eta(2m), \tag{21}$$
$$\tilde{\eta}(2m+1) = \theta(2m+1). \tag{22}$$

Proof. For $m = 1$ the equalities (19)–(22) are verified directly. We make the induction step from m to $m+1$, assuming that $m \geq 1$.

According to the induction hypothesis we have

$$\tilde{\theta}(2m+2) = \tilde{\theta}(2m+1) + \tilde{\eta}(2m+1) = \eta(2m+1) + \theta(2m+1) = \theta(2m+2). \tag{23}$$

The relation (19) is established. The relation (21) is verified in a similar manner:

$$\begin{aligned} \tilde{\eta}(2m+2) &= \tilde{\theta}(2m+1) + [2^{2m+1} - \tilde{\eta}(2m+1)] = \eta(2m+1) + 2^{2m+1} - \\ &\quad - \theta(2m+1) = 2^{2m+2} - [\theta(2m+1) + (2^{2m+1} - \eta(2m+1))] = \\ &= 2^{2m+2} - \eta(2m+2). \end{aligned} \tag{24}$$

As for the relations (20) and (22), they are verified by using (23) and (24). Indeed,

$$\begin{aligned} \tilde{\theta}(2m+3) &= \tilde{\theta}(2m+2) + \tilde{\eta}(2m+2) = \theta(2m+2) + [2^{2m+2} - \eta(2m+2)] = \\ &= \eta(2m+3); \\ \tilde{\eta}(2m+3) &= \tilde{\theta}(2m+2) + [2^{2m+2} - \tilde{\eta}(2m+2)] = \theta(2m+2) + 2^{2m+2} - \\ &\quad - [2^{2m+2} - \eta(2m+2)] = \theta(2m+3). \end{aligned}$$

The theorem is proved. □

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