# FURTHER PROPERTIES OF THE SHAPIRO POLYNOMIALS*

V. N. Malozemov I. S. Stojanoska
malv@math.spbu.ru irena.stoyanoska@gmail.com

28 January 2012

This paper is a continuation of the work given in [1].

**1**°. Recall the definition of the Shapiro polynomials

$$P_0(x) \equiv 1, \quad Q_0(x) \equiv 1$$

and when $m = 0, 1, \dots$

$$\begin{aligned} P_{m+1} &= P_m(x) + x^{2^m} Q_m(x), \\ Q_{m+1} &= P_m(x) - x^{2^m} Q_m(x). \end{aligned} \tag{1}$$

The polynomial $Q_m(x)$ is called *supplementary* to the polynomial $P_m(x)$.

Let

$$P_m(x) = \sum_{k=0}^{2^m-1} a_k\, x^k.$$

The following formula for the coefficients $a_k$ is given in [1, 2]

$$a_k = (-1)^{\sum_{\alpha=1}^{m-1} k_{\alpha-1} k_\alpha}, \quad k \in 0 : 2^m - 1,\ m \geq 2, \tag{2}$$

where $k = (k_{m-1}, k_{m-2}, \dots, k_0)_2$, $k_\alpha \in \{0, 1\}$.

Denote by $b_k$ the coefficients of the polynomial $Q_m(x)$

$$Q_m(x) = \sum_{k=0}^{2^m-1} b_k\, x^k.$$

*Seminar on Discrete Harmonic Analysis and Computer Aided Geometric Design (DHA & CAGD): `http://www.dha.spb.ru/`

**THEOREM 1.** *For $m \geq 1$ the following equality holds*

$$b_k = (-1)^{k_{m-1}} a_k, \quad k \in 0 : 2^m - 1. \tag{3}$$

P r o o f. According to (1) for $m \geq 1$ we have

$$Q_m(x) = P_{m-1}(x) - [P_m(x) - P_{m-1}(x)].$$

It follows from here that

$$\begin{aligned} b_k &= a_k \quad \text{for } k \in 0 : 2^{m-1} - 1, \\ b_k &= -a_k \quad \text{for } k \in 2^{m-1} : 2^m - 1. \end{aligned} \tag{4}$$

The formula (3) is a union of these relations. □

Based on the formulas (2) and (3) we obtain an explicit representation of the coefficients $b_k$ when $m \geq 2$:

$$b_k = (-1)^{\sum_{\alpha=1}^{m-1} k_{\alpha-1} k_\alpha + k_{m-1}}, \quad k \in 0 : 2^m - 1.$$

Note also, that the vectors $a = (a_0, a_1, \ldots, a_{2^m-1})$ and $b = (b_0, b_1, \ldots, b_{2^m-1})$ — coefficients of the polynomials $P_m(x)$ и $Q_m(x)$ are orthogonal. Indeed, by (3)

$$\begin{aligned} \langle a, b \rangle &= \sum_{k=0}^{2^m-1} a_k b_k = \sum_{k=0}^{2^m-1} (-1)^{k_{m-1}} a_k^2 = \\ &= \Bigg( \sum_{k=0}^{2^{m-1}-1} + \sum_{k=2^{m-1}}^{2^m-1} \Bigg) (-1)^{k_{m-1}} = 0. \end{aligned}$$

**2°**. We will need one additional property of the coefficients of the Shapiro polynomials $P_m(x)$.

**LEMMA 1.** *For $m \geq 2$ the following relations hold*

$$a_{2^m-1-k} = (-1)^{m+1+k} a_k, \quad k \in 0 : 2^{m-1} - 1; \tag{5}$$

$$a_{2^m-1-k} = (-1)^{m+k} a_k, \qquad k \in 2^{m-1} : 2^m - 1. \tag{6}$$

P r o o f. When $m = 2$ the equalities (5) and (6) are verified directly. We assume that $m \geq 3$.

We turn first to the equality (5). Let $k \in 0 : 2^{m-1} - 1$, $k = (k_{m-2}, \ldots, k_0)_2$. According to (2)

$$a_k = (-1)^{\sum_{\alpha=1}^{m-2} k_{\alpha-1} k_\alpha}.$$

Further,

$$k' := 2^m - 1 - k = (1,\, 1-k_{m-2}, \dots, 1-k_0)_2.$$

(To check this equality one can use the following fact

$$k + k' = 2^m - 1 = (1, \dots, 1)_2.)$$

According to formula (2)

$$\begin{aligned} a_{k'} &= (-1)^{\sum_{\alpha=1}^{m-2}(1-k_{\alpha-1})(1-k_\alpha)+1-k_{m-2}} = \\ &= (-1)^{m-1-k_0-2\sum_{\alpha=1}^{m-2}k_\alpha+\sum_{\alpha=1}^{m-2}k_{\alpha-1}k_\alpha} = (-1)^{m+1+k}a_k. \end{aligned}$$

The equality (5) is established.

The equality (6) is derived using (5). Let $k \in 2^{m-1} : 2^m - 1$. In this case, the index $k' = 2^m - 1 - k$ belongs to the set $0 : 2^{m-1} - 1$. According to (5) we have

$$a_{2^m-1-k'} = (-1)^{m+1+k'}a_{k'}$$

or

$$a_{2^m-1-k} = (-1)^{m+k}a_k.$$

The lemma is proved. □

**3°**. We now consider the polynomials Shapiro as functions of the complex variable $z$. Of particular interest is the case when $|z| = 1$.

The following result was obtained by H. Shapiro in 1951 [3].

**THEOREM 2.** *For $m \geq 1$ the following identities hold*

$$P_m(-z) = (-1)^m z^{2^m-1} Q_m(\tfrac{1}{z}), \tag{7}$$

$$Q_m(-z) = (-1)^{m+1} z^{2^m-1} P_m(\tfrac{1}{z}). \tag{8}$$

Usually these relations are proved by induction. We will give direct proof.

We start with formula (7). When $m = 1$ its validity can be verified directly. We assume that $m \geq 2$.

According to (4) we have

$$\begin{aligned} (-1)^m z^{2^m-1} Q_m(\tfrac{1}{z}) &= (-1)^m \sum_{k=0}^{2^m-1} b_k\, z^{2^m-1-k} = \\ = (-1)^m &\Bigg[ \sum_{k=0}^{2^{m-1}-1} a_k\, z^{2^m-1-k} - \sum_{k=2^{m-1}}^{2^m-1} a_k\, z^{2^m-1-k} \Bigg]. \end{aligned}$$

At the same time, according to (6)

$$\sum_{k=0}^{2^{m-1}-1} a_k\, z^{2^m-1-k} = \sum_{k=2^{m-1}}^{2^m-1} a_{2^m-1-k}\, z^k = \sum_{k=2^{m-1}}^{2^m-1} (-1)^{m+k} a_k\, z^k$$

and according to (5)

$$-\sum_{k=2^{m-1}}^{2^m-1} a_k\, z^{2^m-1-k} = -\sum_{k=0}^{2^{m-1}-1} a_{2^m-1-k}\, z^k = \sum_{k=0}^{2^{m-1}-1} (-1)^{m+k} a_k\, z^k.$$

Therefore,

$$(-1)^m z^{2^m-1}\, Q_m(\tfrac{1}{z}) = \sum_{k=0}^{2^m-1} (-1)^k a_k\, z^k = P_m(-z).$$

The validity of the identity (7) is established.

Substitute in (7) $-\frac{1}{z}$ instead of $z$. We obtain

$$P_m(\tfrac{1}{z}) = (-1)^m(-\tfrac{1}{z})^{2^m-1} Q_m(-z) = (-1)^{m+1} z^{-2^m+1} Q_m(-z),$$

which is equivalent to (8).

The theorem is proved. □

**4°**. Let us consider one elegant property of the Shapiro polynomials' coefficients, noted in [2].

Denote by $\theta(m)$ the number of positive coefficients of $P_m(x)$.

**THEOREM 3.** *For $m \geq 0$ the following equalities hold*

$$\theta(2m) = 2^{2m-1} + 2^{m-1}, \tag{9}$$

$$\theta(2m+1) = 2^{2m} + 2^m. \tag{10}$$

We present a detailed p r o o f of this assertion.

When $m = 0$ the formulas (9) and (10) take the form $\theta(0) = 1$, $\theta(1) = 2$. Their validity is obvious.

Let us write for $m \geq 0$ the expression for $P_m(x)$,

$$P_m(x) = \sum_{k=0}^{2^m-1} a_k\, x^k,$$

and for $P_{m+1}(x)$ (see [1]),

$$P_{m+1}(x) = \sum_{k=0}^{2^m-1} a_k\, x^{2k} + \sum_{k=0}^{2^m-1} (-1)^k a_k\, x^{2k+1}. \tag{11}$$

Denote by $\theta_0(m)$ the number of positive $a_k$ with even indices and by $\theta_1(m)$ — the number of positive $a_k$ with odd indices. It is obvious that $\theta(m) = \theta_0(m) + \theta_1(m)$. Furthermore, according to (11)

$$\theta_0(m+1) = \theta(m) \quad \text{for } m \geq 0.$$

The last relation can be rewritten as

$$\theta_0(m) = \theta(m-1) \quad \text{for } m \geq 1. \tag{12}$$

From (11) and (12) it follows that for $m \geq 1$

$$\begin{aligned} \theta(m+1) = \theta(m) + \theta_0(m) + [2^{m-1} - \theta_1(m)] = \\ = 2\theta_0(m) + 2^{m-1} = 2\theta(m-1) + 2^{m-1}. \end{aligned}$$

We obtain the recurrence relation

$$\begin{gathered} \theta(m+1) - 2\theta(m-1) = 2^{m-1}, \quad m = 1, 2, \ldots; \\ \theta(0) = 1, \quad \theta(1) = 2. \end{gathered} \tag{13}$$

From (13) we can obtain an explicit expression for $\theta(m)$.

The difference equation (13) has a simple particular solution $\hat{\theta}(m) = 2^{m-1}$. We will seek the solution of the homogeneous equation $\theta(m+1) - 2\theta(m-1) = 0$ in the form $\theta(m) = \lambda^{m+1}$. The value of $\lambda$ (more precisely two of its values) are determined from the condition $\lambda^2 - 2 = 0$. We arrive at the general solution of the equation (13)

$$\theta(m) = 2^{m-1} + c_1(\sqrt{2})^{m+1} + c_2(-\sqrt{2})^{m+1}.$$

We choose the constants $c_1$ и $c_2$ such that the initial conditions are satisfied

$$\begin{aligned} &1 = \theta(0) = \tfrac{1}{2} + c_1\sqrt{2} - c_1\sqrt{2}, \\ &2 = \theta(1) = 1 + 2c_1 + 2c_2. \end{aligned}$$

We obtain

$$c_1 = \frac{2+\sqrt{2}}{8}, \quad c_2 = \frac{2-\sqrt{2}}{8}.$$

The explicit expression for $\theta(m)$ takes the form

$$\theta(m) = 2^{m-1} + \frac{2+\sqrt{2}}{8}(\sqrt{2})^{m+1} + \frac{2-\sqrt{2}}{8}(-\sqrt{2})^{m+1}.$$

This obviously implies the equalities (9) and (10).

The theorem is proved. □

**COROLLARY.** *For $m \ge 0$ the following equalities hold*

$$P_{2m}(1) = 2^m, \qquad P_{2m+1}(1) = 2^{m+1},$$
$$P_{2m}(-1) = 2^m, \quad P_{2m+1}(-1) = 0.$$

When $m = 0$ the equalities are obvious. For $m \ge 1$ we take into account that

$$P_m(1) = \sum_{k=0}^{2^m-1} a_k = \theta(m) - [2^m - \theta(m)] = 2\theta(m) - 2^m,$$

$$\begin{aligned} P_m(-1) = \sum_{k=0}^{2^m-1} (-1)^k a_k &= \theta_0(m) - [2^{m-1} - \theta_0(m)] - \\ &- \{\theta_1(m) - [2^{m-1} - \theta_1(m)]\} = 2\,[\theta_0(m) - \theta_1(m)] = \\ &= 2\,[\,2\theta_0(m) - \theta(m)] = 2\,[\,2\theta(m-1) - \theta(m)]. \end{aligned}$$

## REFERENCES

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