h-POLYHEDRAL SEPARABILITY AND LINEAR PROGRAMMING*

E. K. Cherneutsanu

katerinache@yandex.ru

January 29, 2011

1°. Suppose we have two finite sets in $\mathbb{R}^n$

$$A = \{a_i\}_{i=1}^m \quad \text{and} \quad B = \{b_j\}_{j=1}^k.$$

In the papers [1, 2] a problem of a strict separation of the sets A and B by using h hyperplanes was considered in case of $\operatorname{conv}(A) \cap B = \varnothing$. It was found that this problem could be reduced to the following extremal problem:

$$F(G) := \frac{1}{m} \sum_{i=1}^{m} \varphi_i(G) + \frac{1}{k} \sum_{j=1}^{k} \psi_j(G) \to \inf, \tag{1}$$

where

$$\varphi_i(G) = \max_{s \in 1:h} \big[\langle w^s, a_i \rangle - \gamma_s + 1\big]_+,$$
$$\psi_j(G) = \min_{s \in 1:h} \big[-\langle w^s, b_j \rangle + \gamma_s + 1\big]_+.$$

Unknown is the $\big(h \times (n+1)\big)$-matrix G with rows (w^s, γ_s), $s \in 1:h$.

It is clear that $F(G) \geq 0$ for all G. Condition $F(G_*) = 0$ describes the situation of strict h-separation.

In this report we show that the problem (1) is *equivalent* to a finite number of linear programming problems.

*Seminar on Discrete Harmonic Analysis and Computer Aided Geometric Design (DHA & CAGD): `http://www.dha.spb.ru/`

2°. Denote $\Pi = \big\{S = (s_1, \dots, s_k) \mid s_j \in 1:h,\ j \in 1:k\big\}$.

LEMMA. *The following equality holds*

$$\inf_G F(G) = \min_{S\in\Pi} \inf_G \Big\{ \frac{1}{m} \sum_{i=1}^{m} \varphi_i(G) + \frac{1}{k} \sum_{j=1}^{k} \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+ \Big\}. \tag{2}$$

P r o o f. At first we will establish the formula

$$\sum_{j=1}^{k} \psi_j(G) = \min_{S\in\Pi} \sum_{j=1}^{k} \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+. \tag{3}$$

Let $\psi_j(G) = \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+$ for $j \in 1:k$. Denote $S = (s_1, \dots, s_k)$. Then

$$\sum_{j=1}^{k} \psi_j(G) = \sum_{j=1}^{k} \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+ \geq \min_{S\in\Pi} \sum_{j=1}^{k} \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+. \tag{4}$$

On the other hand, for $S \in \Pi$ we have

$$\sum_{j=1}^{k} \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+ \geq \sum_{j=1}^{k} \min_{s\in 1:h} \big[-\langle w^{s}, b_j\rangle + \gamma_{s} + 1\big]_+ = \sum_{j=1}^{k} \psi_j(G). \tag{5}$$

Taking the left side of (5) to the minimum over $S \in \Pi$, we obtain the inequality that is opposite to (4). The equality (3) is set.

From (3) it follows that

$$F(G) = \min_{S\in\Pi} \Big\{ \frac{1}{m} \sum_{i=1}^{m} \varphi_i(G) + \frac{1}{k} \sum_{j=1}^{k} \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+ \Big\}$$

and

$$\inf_G F(G) = \inf_G \min_{S\in\Pi} \Big\{ \frac{1}{m} \sum_{i=1}^{m} \varphi_i(G) + \frac{1}{k} \sum_{j=1}^{k} \big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+ \Big\}. \tag{6}$$

All that remains is to interchange the infimum over G and the minimum over $S \in \Pi$ in the right side of (6). The lemma is proved. □

3°. The lemma shows that the problem (1) is equivalent to a finite number of extremal problems of a kind

$$\frac{1}{m}\sum_{i=1}^{m}\max_{s\in 1:h}\big[\langle w^s, a_i\rangle - \gamma_s + 1\big]_+ + \frac{1}{k}\sum_{j=1}^{k}\big[-\langle w^{s_j}, b_j\rangle + \gamma_{s_j} + 1\big]_+ \to \inf_G, \tag{7}$$

corresponding to different $S \in \Pi$. In turn, the problem (7) is equivalent to a linear programming problem

$$\frac{1}{m}\sum_{i=1}^{m} p_i + \frac{1}{k}\sum_{j=1}^{k} q_j \to \inf, \tag{8}$$
$$-\langle a_i, w^s\rangle + \gamma_s + p_i \geq 1, \quad i \in 1:m,\ s \in 1:h;$$
$$\langle b_j, w^{s_j}\rangle - \gamma_{s_j} + q_j \geq 1, \quad j \in 1:k;$$
$$p_i \geq 0, \quad i \in 1:m; \quad q_j \geq 0, \quad j \in 1:k.$$

We come to the following conclusion.

THEOREM 1. *The problem* (1) *is equivalent to a finite number of linear programming problems of a kind* (8) *in a sense that the solution of the problem* (8) *with* $S \in \Pi$ *that corresponds to the smallest objective function value is a solution of the problem* (1).

4°. Consider an example. Let the sets A and B on a plane are given which consist of the points

$$a_1 = (-2, 0),\ a_2 = (2, 0),\ a_3 = (0, 2),\ a_4 = (0, 1),$$

and

$$b_1 = (0, 3),\ b_2 = (3, 0),\ b_3 = (-3, 0),$$

respectively. It is obvious that $\mathrm{conv}(A) \cap B = \varnothing$ (see Figure 1).

We will solve the problem of 2-polyhedral separation. In this case,

$$n = 2,\ m = 4,\ k = 3,\ h = 2.$$

Let us find out how does the problem (8) look like if $S = (1,\, 1,\, 2)$. We write the vector of unknowns

$$z = (w_1^1,\, w_2^1,\, \gamma_1,\, w_1^2,\, w_2^2,\, \gamma_2,\, p_1,\, p_2,\, p_3,\, p_4,\, q_1,\, q_2,\, q_3)$$

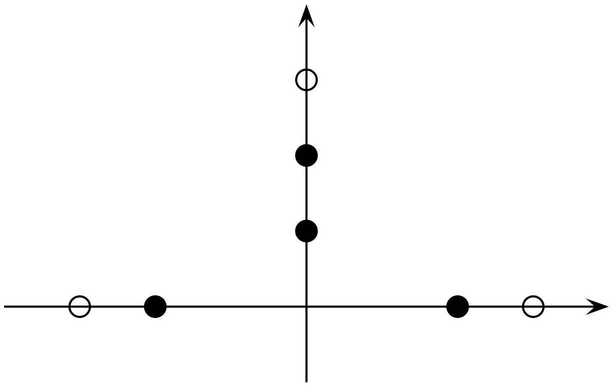

Figure 1

and the matrix of constraints

$$D = \begin{bmatrix}
2 & 0 & 1 & & & & 1 & & & & & & \\
-2 & 0 & 1 & & & & & 1 & & & & & \\
0 & -2 & 1 & & & & & & 1 & & & & \\
0 & -1 & 1 & & & & & & & 1 & & & \\
 & & & 2 & 0 & 1 & 1 & & & & & & \\
 & & & -2 & 0 & 1 & & 1 & & & & & \\
 & & & 0 & -2 & 1 & & & 1 & & & & \\
 & & & 0 & -1 & 1 & & & & 1 & & & \\
0 & 3 & -1 & & & & & & & & 1 & & \\
3 & 0 & -1 & & & & & & & & & 1 & \\
 & & & -3 & 0 & -1 & & & & & & & 1
\end{bmatrix}.$$

The problem (8) takes the form

$$\frac{1}{4}\sum_{i=1}^{4} p_i + \frac{1}{3}\sum_{j=1}^{3} q_j \to \inf, \tag{9}$$
$$Dz \geq e,$$
$$p_i \geq 0, \quad i \in 1:4; \qquad q_j \geq 0, \quad j \in 1:3.$$

The solution of this problem is a vector

$$z = (111.2210,\ 112.0012,\ 272.9097,\ -78.1474,\ 27.3511,\ 192.1601,$$
$$0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000).$$

The minimum objective function value is zero, 2-polyhedral separation of the sets A and B in case of $S = (1, 1, 2)$ is shown on Figure 2.

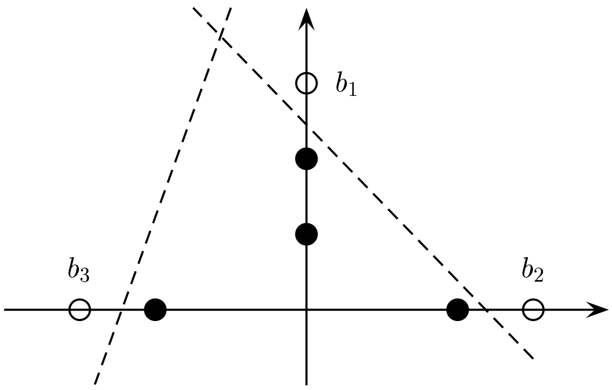

Figure 2

Note that setting the vector of indices S to (1, 1, 2) corresponds to partitioning the set B into two subsets $\{b_1, b_2\} \cup \{b_3\}$. These two subsets are concordantly separated from the set A with two straight lines

$$\langle w^1, x\rangle = \gamma_1 \quad \text{and} \quad \langle w^2, x\rangle = \gamma_2.$$

There are two more ways to partition the set B into two subsets:

$$\{b_1, b_3\} \cup \{b_2\} \quad \text{and} \quad \{b_2, b_3\} \cup \{b_1\}.$$

They correspond to the vectors $S = (1, 2, 1)$ and $S = (2, 1, 1)$.

The result of 2-polyhedral separation in case of $S = (1, 2, 1)$ is shown on Figure 3.

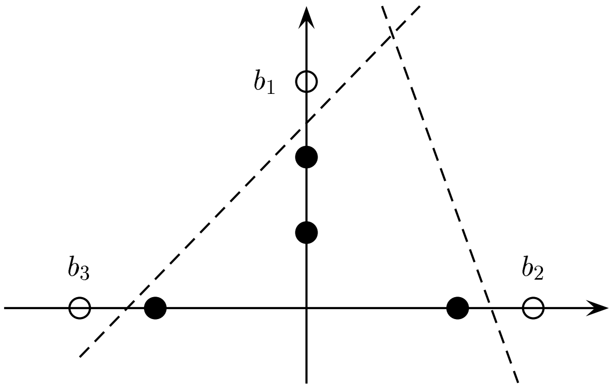

Figure 3

This case is symmetric to the case $S = (1, 1, 2)$.

In case of $S = (2, 1, 1)$ there is no 2-polyhedral separation (see Figure 4).

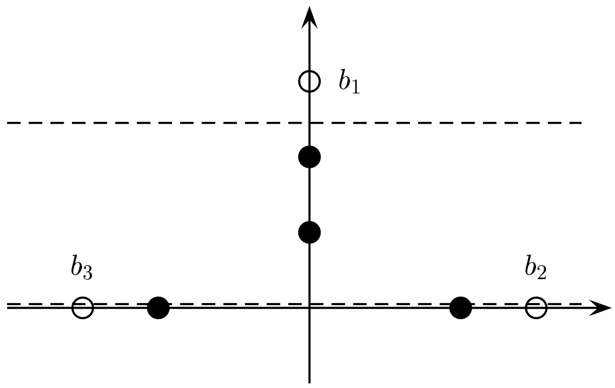

Figure 4

A solution of the problem similar to (9) is a vector

$$z = (0.0000,\ -112.0230,\ -1.0000,\ 0.0000,\ 111.8673,\ 273.7824,\\ 2.0000,\ 2.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000).$$

The minimum objective function value equals to one.

5°. In general, the assignment of the vector $S \in \Pi$ corresponds to partitioning the set B that consists of k vectors into h subsets. It is the number of such partitions that determines the number of linear programming problems of a kind (8) that the solution of the problem (1) is being reduced to.

If you know beforehand that the sets A and B are h–polyhedral separable, the solution of the problem (1) can be simplified. After partitioning the set B into h subsets, you should *independently* solve the problems of linear separation of each of these subsets from the set A. In case of a successful separation, the collection of separating hyperplanes forms a solution of the problem (1).

In the above example, when $S = (1, 1, 2)$, we will independently solve the problems of linear separation of the sets $\{b_1,\, b_2\}$ and $\{b_3\}$ from A. Let us write the corresponding linear programming problems (cf. [3])

$$\begin{gathered}
\frac{1}{4}\sum_{i=1}^{4} p_i + \frac{1}{2}\sum_{j=1}^{2} q_j \to \inf, \\
-\langle a_i, w^1\rangle + \gamma_1 + p_i \geq 1, \quad i \in 1:4; \\
\langle b_j, w^1\rangle - \gamma_1 + q_j \geq 1, \quad j \in 1:2; \\
p_i \geq 0, \quad i \in 1:4; \qquad q_j \geq 0, \quad j \in 1:2,
\end{gathered}$$

and

$$\frac{1}{4}\sum_{i=1}^{4} p_i + q_3 \to \inf,$$
$$-\langle a_i, w^2\rangle + \gamma_2 + p_i \geq 1, \quad i \in 1:4;$$
$$\langle b_3, w^2\rangle - \gamma_2 + q_3 \geq 1;$$
$$p_i \geq 0, \quad i \in 1:4; \qquad q_3 \geq 0.$$

Their solutions $\{w^1, \gamma_1\}$ and $\{w^2, \gamma_2\}$ define two straight lines that strictly separate A from B (see Figure 5).

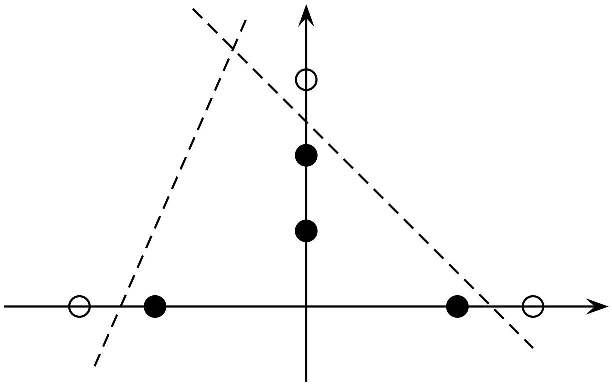

Figure 5

REFERENCES

1. Astorino A., Gaudioso M. *Polyhedral separability througt successive LP* // JOTA. 2002. Vol. 112. No. 2. P. 265–293.

2. Malozemov V. N., Cherneutsanu E. K. *Strict h-separability of two sets* // Seminar «DHA & CAGD». Selected Reports. December 18, 2010. (`http://dha.spb.ru/reps10.shtml#1218`)

3. Malozemov V. N., Cherneutsanu E. K. *On mathematical diagnostics (linear model)* // Seminar «DHA & CAGD». Selected Reports. April 17, 2010. (`http://dha.spb.ru/reps10.shtml#0417`)